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A projection-valued measure for a countable iterated function system

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Abstract Davison in 2015 used the famous Banach Fixed Point Theorem to prove that a certain class of iterated function systems generated counterparts of the Hutchinson measure in the space of projection-valued measures. In this paper, we generalize this result by considering iterated function systems with infinitely many maps.

Keywords Kantorovich metric · Projection-valued measure · Countable iterated function system · Fixed point

Mathematics Subject Classification (2010) 46C99 · 37C25

1 Introduction

Let (Y, d) be a compact metric space. A map L from Y into itself is a *Lipschitz contraction* on (Y, d) if there exists a constant c, $0 < c < 1$, such that

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$d(L(x), L(y)) \leq cd(x, y)$, for all $x, y \in Y$. The smallest of all such constants is called *Lipschitz constant of* L. Let L be a Lipschitz contraction on (Y, d). Since Y is a complete metric space, it is well known that L admits a unique fixed point $y \in Y$, meaning that $L(y) = y$. This result is known as the Contraction Mapping Principle or the Banach Fixed Point Theorem.

In [6], Hutchinson generalized the Banach Fixed Point Theorem to a finite family $\mathcal{S} = \{\sigma_1, \dots, \sigma_N\}$ of Lipschitz contractions on (Y, d). Precisely, he proved that there is a unique compact subset $X \subset Y$ which is invariant under $\mathcal{S}$, meaning that

$$X = \bigcup_{i=1}^{N} \sigma_i(X).$$

A finite family of Lipschitz contractions on (Y, d) is called an *iterated function system* (IFS) on Y, and the compact invariant subset X described above is called the *self-similar fractal set*, or *attractor set*, associated to the IFS. Moreover, Hutchinson showed that the attractor set can be realized as the support of a Borel probability measure on Y. This measure, which we denote by μ, satisfies the fixed point relation

$$\mu(\cdot) = \sum_{i=1}^{N} \frac{1}{N} \mu(\sigma_i^{-1}(\cdot)),$$

and is often referred to as the *Hutchinson measure*. It is the unique fixed point of an appropriate Lipschitz contraction on the complete metric space of Borel probability measures on Y equipped with the classical Kantorovich metric H given by

$$H(\mu, \nu) = \sup_{f \in Lip_1(Y)} \left\{ \left| \int_Y f d\mu - \int_Y f d\nu \right| \right\}, \tag{1}$$

where $Lip_1(Y) = \{f : Y \to \mathbb{R} : |f(x) - f(y)| \leq d(x, y) \text{ for all } x, y \in Y\}$.

In [7,8], Jorgensen generalized the Hutchinson measure to operator-valued measures. He considered the Hilbert space $L^2(X, \mu)$, where $X \subset Y$ is the attractor associated to the IFS and μ is the Hutchinson measure on Y, and showed that there exists a unique projection-valued measure, E, defined on the Borel sigma algebra of X taking values in the projections on $L^2(X, \mu)$ such that

$$E(\cdot) = \sum_{i=1}^{N} S_i E(\sigma_i^{-1}(\cdot)) S_i^*, \tag{2}$$

for certain isometries S_i on $\mathcal{H}$ and their adjoints S_i^*.

In [4,5], Davison developed an alternative approach to proving this result. In particular, given the Hilbert space $\mathcal{H} = L^2(X, \mu)$ (or more generally, a Hilbert space $\mathcal{H}$ which admits a representation of the Cuntz algebra on N generators), he considered the space of projection-valued measures from the Borel sigma algebra of X into the projections on $\mathcal{H}$, and showed that this space can be made into a complete an bounded metric space via a generalized

Kantorovich metric. Davison used this result and the Fixed Point Theorem to prove that there exists a unique projection-valued measure E satisfying (2).

The notions of the attractor set and the Hutchinson measure can be generalized to a *countable iterated function system* (CIFS) (see [1, 9]). Precisely, given a countable family $\mathcal{S} = \{\sigma_i : i \in \mathbb{N}\}$ of Lipschitz contractions on a compact metric space (Y, d) such that $\sup\{r_i : i \in \mathbb{N}\} < 1$, where r_i is the Lipschitz constant associated to σ_i, there exists a unique compact invariant set $X \subset Y$ such that

$$X = \overline{\bigcup_{i\in\mathbb{N}} \sigma_i(X)}. \tag{3}$$

This unique invariant set said to be the *attractor set* associated to the CIFS. Furthermore, if $\mathcal{P} = \{\rho_i\}_{i\in\mathbb{N}}$ is a probability sequence, then there exists a unique invariant Borel probability measure μ on Y, called the *Generalized Hutchinson measure* associated to $(\mathcal{S}, \mathcal{P})$, such that

$$\mu(\cdot) = \sum_{i=1}^{\infty} \rho_i \mu \circ {\sigma_i}^{-1}(\cdot), \tag{4}$$

and $supp(\mu) = X$ (see [2]). Note that Bandt [1] showed that the attractor set associated to the CIFS is not necessarily compact provide that (Y, d) not is a compact metric space.

The main goal of this paper is to study a generalization of above result to projection-valued measures. Although the techniques used in our proofs are similar than those used in [4], the iterated function system with countably many of Lipschitz contractions setting requires highly more effort.

The rest of the paper is structured as follows. In Section 2, we give all assumptions and preliminary concepts which we need later. In Section 3, we consider a CIFS, say $\mathcal{S} = \{\sigma_i : i \in \mathbb{N}\}$, a probability sequence $\mathcal{P}$ and the Hilbert space $L^2(X, \mu)$, where $X \subset Y$ is the attractor associated to $\mathcal{S}$ and μ is the Generalized Hutchinson measure associated to $(\mathcal{S}, \mathcal{P})$. Then, show that there exists a map on the space of projection-valued measures from the Borel sigma algebra of X into the projections on $L^2(X, \mu)$. In Section 4, we prove that the aforementioned map is a Lipschitz contraction on a complete metric space via the generalized Kantorovich metric. As a consequence, we see that there exists a unique projection-valued measure for $\mathcal{S}$ such that

$$E(\cdot) = \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\cdot)) S_i^*,$$

for certain isometries S_i on $L^2(X, \mu)$ and their adjoints S_i^*.

2 Preliminaries

In this section we recall assumptions and preliminary concepts will be needed throughout the paper.

Let (X, d) be a compact metric space and let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an arbitrary Hilbert space. We denote by $\mathcal{B}(X)$ the Borel sigma algebra of X and by $B(\mathcal{H})$ space of bounded linear operators on $\mathcal{H}$ which are orthogonal projections (i.e., self-adjoint and idempotent operators).

Definition 1 A projection-valued measure with respect to the pair $(X, \mathcal{H})$ is a map $F : \mathcal{B}(X) \to B(\mathcal{H})$ such that:

1. $F(\Delta)$ is a projection in $B(\mathcal{H})$ for all $\Delta \in \mathcal{B}(X)$;
2. $F(\emptyset) = 0$ and $F(X) = id_{\mathcal{H}}$ (the identity operator on $\mathcal{H}$);
3. $F(\Delta_1 \cap \Delta_2) = F(\Delta_1)F(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{B}(X)$ (where the product operation $F(\Delta_1)F(\Delta_2)$ is the operator composition in $B(\mathcal{H})$);
4. If $\{\Delta_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets in $\mathcal{B}(X)$, and if $\phi, \psi \in \mathcal{H}$, then
$$\left\langle F\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right)\phi, \psi\right\rangle = \sum_{n=1}^{\infty} \langle F(\Delta_n)\phi, \psi\rangle .$$

Lemma 1 *[3, Lemma 1.9, p. 257] Let E be a projection-valued measure with respect to the pair $(X, \mathcal{H})$. For all $\phi, \psi \in \mathcal{H}$ and $\Delta \in \mathcal{B}(X)$,*
$$E_{\phi,\psi}(\Delta) = \langle E(\Delta)\phi, \psi\rangle$$
defines a countably additive measure on $\mathcal{B}(X)$ with total variation less than or equal to $\|\phi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}}$. Moreover, $E_{\phi,\psi}(\cdot) = \overline{E_{\psi,\phi}(\cdot)}$.

Remark 1 If $\phi \in \mathcal{H}$, $E_{\phi,\phi}(\cdot)$ is a positive measure with total mass equal to $\|\phi\|^2$.

Let $P(X)$ be the space of projection-valued measures from $\mathcal{B}(X)$ into the projections on $\mathcal{H}$. Define the *generalized Kantorovich* metric ρ on $P(X)$ by
$$\rho(E, F) = \sup_{f\in Lip_1(X)} \left\{ \left\| \int f dE - \int f dF \right\| \right\} \tag{5}$$
where $\|\cdot\|$ denotes the operator norm in $B(\mathcal{H})$, E and F are arbitrary members of $P(X)$, and $\int f dE$ is the unique bounded operator on $\mathcal{H}$ that satisfies
$$\left\langle \left(\int f dE\right)\phi, \psi\right\rangle = \int_X f dE_{\phi,\psi} \quad \text{for all} \quad \phi, \psi \in \mathcal{H}.$$

Recently, T. Davison [4] proved the following property of the metric space $(P(X), \rho)$.

Theorem 1 *[4, Theorem 2.11] Let (X, d) be a compact metric space. Then the metric space $(P(X), \rho)$ is complete.*

Let $\mathcal{S} = \{\sigma_i : i \in \mathbb{N}\}$ be a CIFS on (Y, d) with Lipschitz constants r_i. Assume $r := \sup\{r_i : i \in \mathbb{N}\} < 1$ and let $X \subset Y$ be the attractor associated to the CIFS satisfying $\sigma_i(X) \cap \sigma_j(X) = \emptyset$ for $i \neq j$.

We denote by $M(X)$ the space of Borel probability measures on X.

From now on, $\mathcal{P} = \{\rho_i\}_{i\in\mathbb{N}}$ is a probability sequence and $\mu \in M(X)$ is the Generalized Hutchinson measure associated to $(\mathcal{S}, \mathcal{P})$. Additionally, $\sigma : X \to X$ is a Borel measurable function such that

$$(\sigma \circ \sigma_j)(x) = x, \text{ for all } x \in \bigcup_{i\in\mathbb{N}} \sigma_i(X), \quad j \in \mathbb{N}. \tag{6}$$

As

$$\mu\left(X \setminus \bigcup_{i\in\mathbb{N}} \sigma_i(X)\right) = 1 - \sum_{i=1}^{\infty} \rho_i \mu(\sigma_i^{-1}(\sigma_i(X))) = 0, \tag{7}$$

we have

$$\sigma \circ \sigma_j = id_X, \quad j \in \mathbb{N}. \tag{8}$$

The following theorem shows a simple manner to generate a CIFS from a IFS, under the above hypotheses.

Theorem 2 *Let $\mathcal{S} = \{\tau_1, \ldots, \tau_n\}$ be a IFS of injective maps on (Y, d) such that the attractor set X associated to $\mathcal{S}$ satisfies*

$$\tau_i(X) \cap \tau_j(X) = \emptyset \quad \textit{for} \quad i \neq j. \tag{9}$$

Let $\mathcal{F} = \{\sigma_i : i \in \mathbb{N}\}$ be the family given by $\sigma_i = \tau_n^p \circ \tau_{q+1}$ if $i - 1$ is of the form $p(n-1) + q$ with $p \in \mathbb{N} \cup \{0\}$ and $0 \leq q < n - 1$. Then $\mathcal{F}$ is a CIFS on (Y, d) such that

(a) Lipschitz constants s_i associated to σ_i verifies $\sup\{s_i : \sigma_i \in \mathcal{F}\} < 1$,
(b) $\sigma_i(X) \cap \sigma_j(X) = \emptyset$ for $i \neq j$,
(c) the attractor set associated to $\mathcal{F}$ is X.

Proof It is easy to see that $\mathcal{F}$ is a CIFS on (Y, d) satisfying (a).
Now assume $i \neq j$ and let $i - 1 = p(n-1) + q$ and $j - 1 = p'(n-1) + q'$ with $p, p' \in \mathbb{N} \cup \{0\}$ and $0 \leq q, q' < n - 1$. When $p = p'$, $q \neq q'$ and thus

$$\sigma_i(X) \cap \sigma_j(X) = (\tau_n^p \circ \tau_{q+1})(X) \cap (\tau_n^p \circ \tau_{q'+1})(X) = \emptyset,$$

where the last equality is due fact that τ_n^p is a injective map, $q+1 \neq q'+1$, and (9). Suppose $p \neq p'$. Without loss of generality assume $p < p'$. Since $q + 1 < n$, from (9) we have $\tau_{q+1}(X) \cap (\tau_n^{p'-p} \circ \tau_{q'+1})(X) = \emptyset$. Therefore, the injectivity of τ_n^p implies

$$\sigma_i(X) \cap \sigma_j(X) = (\tau_n^p \circ \tau_{q+1})(X) \cap (\tau_n^p \circ \tau_n^{p'-p} \circ \tau_{q'+1})(X) = \emptyset.$$

This completes the proof of (b).
A straightforward computation shows that

$$\overline{\bigcup_{i\in\mathbb{N}} \sigma_i(X)} \subset X,$$

because X is a compact set and $\sigma_i(X) \subset X$, $i \in \mathbb{N}$.
Now, let $x \in X$. By [6, (3) Theorem (iii)], there exists a sequence $\{\alpha_i\}_{i\in\mathbb{N}} \in \{1,\dots,n\}^{\mathbb{N}}$ such that

$$\{x\} = \bigcap_{i\in\mathbb{N}} (\tau_{\alpha_1} \circ \cdots \circ \tau_{\alpha_i})(X).$$

We complete the proof by considering two cases:
(i) $\alpha_i = n$ for all i,
(ii) $\alpha_i < n$ for some i.
Assume (i) and let $y \in X$ be the fixed point of τ_n,

$$M = \max\{d(\tau_{q+1}(y), y) : 0 \le q < n-1\},$$

and $j \in \mathbb{N}$. As $(\tau_{\alpha_1} \circ \cdots \circ \tau_{\alpha_i})(y) = \tau_n^i(y) = y$ for all i, from [6, (3) Theorem (iv)] it follows that $x = y$. Let r_n be the Lipschitz constant associated to τ_n. Since $j-1 = p(n-1)+q$ with $p \in \mathbb{N}\cup\{0\}$ and $0 \le q < n-1$, then

$$d(\sigma_j(y), x) = d((\tau_n^p \circ \tau_{q+1})(y), \tau_n^p(y)) \le r_n^p d(\tau_{q+1}(y), y) \le r_n^p M,$$

and so

$$x = \lim_{j\to\infty} \sigma_j(y) \in \overline{\bigcup_{i\in\mathbb{N}} \sigma_i(X)}.$$

Now suppose (ii) and let $j = \min\{i \in \mathbb{N} : \alpha_i < n\}$. If $j = 1$, then $\tau_{\alpha_1} = \sigma_{\alpha_1}$, and thus

$$x \in \sigma_{\alpha_1}(X) \subset \overline{\bigcup_{i\in\mathbb{N}} \sigma_i(X)}.$$

If $j \ge 2$, then $\tau_{\alpha_1} \circ \cdots \circ \tau_{\alpha_j} = \tau_n^{j-1} \circ \tau_{\alpha_j}$ with $\alpha_j < n$, and therefore

$$x \in (\tau_{\alpha_1} \circ \cdots \circ \tau_{\alpha_j})(X) = \sigma_{(j-1)(n-1)+\alpha_j}(X) \subset \overline{\bigcup_{i\in\mathbb{N}} \sigma_i(X)}.$$

Consequently, X is the attractor set associated to $\mathcal{F}$, and the proof of (c) is complete. □

Remark 2 Note that if $\mathcal{S}$ is a CIFS of injective maps, then every $\sigma_j : (X,d) \to (\sigma_j(X), d)$ is a continuous bijection. So, the function $\sigma : X \to X$ can be constructed as follows: if $x \in \sigma_j(X)$, $j \in \mathbb{N}$, let $\sigma(x) = \sigma_j^{-1}(x)$; otherwise, let $\sigma(x) = x$. Further, it is easy to see that σ is a Borel measurable function. Indeed, as (X,d) is a compact space and $(\sigma_j(X), d)$ is a Hausdorff space, then $\sigma_j : (X,d) \to (\sigma_j(X), d)$ is a homeomorphism. Now, the equality

$$\sigma^{-1}(A) = \left(\left(X \setminus \bigcup_{j\in\mathbb{N}} \sigma_j(X)\right) \cap A\right) \cup \bigcup_{j\in\mathbb{N}} \sigma_j(A), \quad A \subset X,$$

completes the proof.

Next, we offer an example of a possible scenario of work.

Example 1 Let $Y = [0, 1]$ be with the standard metric on $\mathbb{R}$, and we consider the IFS $\{\tau_1, \tau_2\}$ given by

$$\tau_1(x) = \frac{1}{3}x \quad \text{and} \quad \tau_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

It is well known that the Cantor ternary set, say X, is the attractor of $\{\tau_1, \tau_2\}$. Since $\tau_1(X) \cap \tau_2(X) = \emptyset$, from Theorem 2 we obtain that the family $\{\sigma_i : i \in \mathbb{N}\}$ given by

$$\sigma_i(x) = \frac{1}{3^i}x + \frac{3^{i-1} - 1}{3^{i-1}}.$$

is a CIFS satisfying $(a) - (c)$ of Theorem 2.
Finally, the map $\sigma : X \to X$ defined by

$$\sigma(x) = 3^i x - (3^i - 3), \quad x \in \left[\frac{3^{i-1} - 1}{3^{i-1}}, \frac{3^i - 2}{3^i}\right] \cap X,$$

and $\sigma(1) = 1$, is a Borel measurable function satisfying (6).

In the sequel, we consider the Hilbert space $\mathcal{H} = L^2(X, \mu)$, and define $S_i, S_i^* : \mathcal{H} \to \mathcal{H}$, $i \in \mathbb{N}$, by

$$S_i\phi = (\phi \circ \sigma)\frac{1}{\sqrt{\rho_i}}\mathbf{1}_{\sigma_i(X)} \quad \text{and} \quad S_i^*\phi = \sqrt{\rho_i}(\phi \circ \sigma_i), \quad \phi \in \mathcal{H}, \tag{10}$$

where $\mathbf{1}_A$ denotes the characteristic function of a set A.

3 A map on $(P(X), \rho)$

Let $\Phi : P(X) \to P(X)$ be the map given by

$$\Phi(E)(\cdot) = \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\cdot)) S_i^*, \tag{11}$$

where S_i, S_i^*, and σ_i are given in Section 2.

The main aim of this section is to prove that the map Φ is well defined. For this purpose, we need a list of theoretical results which we will show in what follows. First, it is worth remembering the Lebesgue's Monotone Convergence Theorem for Series.

Lemma 2 *[11, p.175] Let $\{a_{ik}\}_{(i,k)\in\mathbb{N}^2}$ be a double sequence of real numbers such that $0 \le a_{ik} \le a_{i(k+1)}$, for all $(i, k) \in \mathbb{N}^2$. Then*

$$\lim_{k\to\infty} \sum_{i=1}^{\infty} a_{ik} = \sum_{i=1}^{\infty} \lim_{k\to\infty} a_{ik}.$$

Next proposition will be useful in what follows.

Proposition 1 *Let $\{\nu_i\}_{i\in\mathbb{N}}$ be a sequence of finite measures on $\mathcal{B}(X)$ and ν be a finite measure on $\mathcal{B}(X)$ such that $\nu(\Delta) = \sum_{i=1}^{\infty} \nu_i(\Delta)$ for all $\Delta \in \mathcal{B}(X)$. If ϕ is $\nu-$integrable function on X, then ϕ is ν_i-integrable function on X for all i, and*

$$\int_X \phi \; d\nu = \sum_{i=1}^{\infty} \int_X \phi \; d\nu_i.$$

Proof It is clear that the proposition holds for characteristic functions. Then, it is also true for non-negative simple functions due to the linearity of the integral. Now, let ϕ be a non-negative ν-measurable function. Then there exists a sequence of non-negative simple functions $\{\phi_k\}$ such that $\phi_k \nearrow \phi$, as $k \to \infty$. According to the Beppo Levi Theorem we have

$$\int_X \phi \; d\nu = \lim_{k\to\infty} \sum_{i=1}^{\infty} \int_X \phi_k \; d\nu_i.$$

Thus, Lemma 2 with $a_{ik} = \int_X \phi_k \; d\nu_i$ implies

$$\int_X \phi \; d\nu = \sum_{i=1}^{\infty} \int_X \phi \; d\nu_i.$$

Finally, the general case of the real and complex functions ϕ also remains valid since $\phi = \phi^+ - \phi^-$ and $\phi = Re(\phi) + iIm(\phi)$, respectively. □

Here below, we show the change of variables formula for integrals.

Theorem 3 *[10, Theorem 1.19] Suppose $\varsigma : X \to X$ is a Borel function, ν is a Borel measure on X and f is a non-negative Borel function on X. Then*

$$\int_{\varsigma(X)} f \; d(\nu \circ \varsigma^{-1}) = \int_X (f \circ \varsigma) \; d\nu,$$

where $\nu \circ \varsigma^{-1}$ is the pushforward measure defined by $(\nu \circ \varsigma^{-1})(\Delta) = \nu(\varsigma^{-1}(\Delta))$, $\Delta \in \mathcal{B}(X)$.

Remark 3 The above theorem remains valid if f is a real $\nu-$integrable function on X.

The following Theorem generalizes [4, Theorem 1.5].

Theorem 4 *The maps S_i, $i \in \mathbb{N}$, are isometries, and the maps S_i^*, $i \in \mathbb{N}$ are their adjoints. Moreover, these maps and their adjoints satisfy the relations:*

(a) $S_i^* S_j = \delta_{ij} id_{\mathcal{H}}$, *for* $i, j \in \mathbb{N}$;
(b) $\sum_{i\in\mathbb{N}} S_i S_i^* = id_{\mathcal{H}}$.

Proof Let $i \in \mathbb{N}$. Clearly, S_i is a linear map. By Proposition 1,

$$\|S_i\phi\|^2_{\mathcal{H}} = \frac{1}{\rho_i}\int_X |(\phi\circ\sigma)\mathbf{1}_{\sigma_i(X)}|^2\, d\mu = \sum_{j=1}^{\infty}\frac{\rho_j}{\rho_i}\int_{\sigma_i(X)} |\phi\circ\sigma|^2 d(\mu\circ\sigma_j^{-1}),$$

Since $\sigma_j^{-1}(\sigma_i(X)) = X$ if $i = j$, and $\sigma_j^{-1}(\sigma_i(X)) = \emptyset$ otherwise, from Theorem 3 and (8), follows

$$\begin{aligned}\|S_i\phi\|^2_{\mathcal{H}} &= \int_{\sigma_i(X)} |\phi\circ\sigma|^2 d(\mu\circ\sigma_i^{-1}) = \int_X |\phi\circ\sigma|^2\circ\sigma_i d\mu\\ &= \int_X |\phi(\sigma\circ\sigma_i)|^2 d\mu = \|\phi\|^2_{\mathcal{H}}.\end{aligned}$$

So, S_i is a isometry.
Now, we show that S_i^* is the adjoint of S_i. Let $\phi, \psi \in \mathcal{H}$. From Proposition 1 and Theorem 3 we have

$$\begin{aligned}\int_X \phi\overline{S_i\psi}d\mu &= \sum_{j=1}^{\infty}\frac{\rho_j}{\sqrt{\rho_i}}\int_{\sigma_i(X)} \phi\overline{\psi\circ\sigma}d(\mu\circ\sigma_j^{-1})\\ &= \frac{\rho_i}{\sqrt{\rho_i}}\int_{\sigma_i(X)} \phi\overline{\psi\circ\sigma}d(\mu\circ\sigma_i^{-1}) = \sqrt{\rho_i}\int_X(\phi\circ\sigma_i)\overline{\psi}d\mu\\ &= \int_X (S_i^*\phi)\overline{\psi}d\mu.\end{aligned}$$

The relation (a) can be easily computed. Indeed, let $\phi \in \mathcal{H}$, then

$$S_i^*S_j\phi = \frac{\sqrt{\rho_i}}{\sqrt{\rho_j}}((\phi\circ\sigma)\mathbf{1}_{\sigma_j(X)})\circ\sigma_i = \frac{\sqrt{\rho_i}}{\sqrt{\rho_j}}(\phi\circ\sigma\circ\sigma_i)(\mathbf{1}_{\sigma_j(X)}\circ\sigma_i),$$

Since $\mathbf{1}_{\sigma_j(X)}\circ\sigma_i = \delta_{i,j}\mathbf{1}_X$, according to (8) we obtain $S_i^*S_j\phi = \delta_{i,j}\phi$.
In order to prove (b), we note that

$$S_iS_i^*\phi = (\phi\circ\sigma_i\circ\sigma)\mathbf{1}_{\sigma_i(X)}.$$

From (6) it follows that $\sigma_i\circ\sigma|_{\sigma_i(X)} = id_{\sigma_i(X)}$, and so $S_iS_i^*\phi = \phi\mathbf{1}_{\sigma_i(X)}$. Thus, (7) shows that

$$\sum_{i=1}^{\infty} S_iS_i^*\phi = \phi\mathbf{1}_{\cup_{i\in\mathbb{N}}\sigma_i(X)} = \phi.$$

The proof is complete. □

We can now show that a series of orthogonal projections can be defined as a orthogonal projection on $\mathcal{H}$.

Lemma 3 *[3, p. 256] Let $\{P_i\}$ be a sequence of pairwise orthogonal projections on $\mathcal{H}$. Then for each ϕ in $\mathcal{H}$, $\sum_{i\in\mathbb{N}} P_i\phi$ converges in $\mathcal{H}$ to $P\phi$, where P is the orthogonal projection of $\mathcal{H}$ onto $\vee\{P_i(\mathcal{H}) :\ i \geq 1\}$.*

Lemma 4 *For each $E \in P(X)$ the map $\Phi(E) : \mathcal{B}(X) \to B(\mathcal{H})$ given by*

$$\Phi(E)(\Delta) = \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta)) S_i^*,$$

is well defined.

Proof Set $E \in P(X)$. For $\Delta \in \mathcal{B}(X)$, let $\{P_i\}$ be a sequence of bounded linear operator on $\mathcal{H}$ given by $P_i = S_i E(\sigma_i^{-1}(\Delta) S_i^*$, $i \in \mathbb{N}$. Let $i \in \mathbb{N}$, $\phi, \psi \in \mathcal{H}$. By Theorem 4 we have

$$\begin{aligned} P_i P_i \phi &= (S_i E(\sigma_i^{-1}(\Delta)) S_i^*)(S_i E(\sigma_i^{-1}(\Delta)) S_i^*)\phi \\ &= S_i E(\sigma_i^{-1}(\Delta)) E(\sigma_i^{-1}(\Delta)) S_i^* \phi \\ &= S_i E(\sigma_i^{-1}(\Delta)) S_i^* \phi \\ &= P_i \phi, \end{aligned}$$

and

$$\begin{aligned} \langle P_i \phi, \psi \rangle &= \left\langle S_i E(\sigma_i^{-1}(\Delta)) S_i^* \phi, \psi \right\rangle \\ &= \left\langle E(\sigma_i^{-1}(\Delta)) S_i^* \phi, S_i^* \psi \right\rangle \\ &= \left\langle S_i^* \phi, E(\sigma_i^{-1}(\Delta)) S_i^* \psi \right\rangle \\ &= \left\langle \phi, S_i E(\sigma_i^{-1}(\Delta)) S_i^* \psi \right\rangle \\ &= \langle \phi, P_i \psi \rangle . \end{aligned}$$

In addition, we also obtain

$$\begin{aligned} \langle P_i \phi, P_j \psi \rangle &= \left\langle S_i E(\sigma_i^{-1}(\Delta)) S_i^* \phi, S_j E(\sigma_j^{-1}(\Delta)) S_j^* \psi \right\rangle \\ &= \left\langle E(\sigma_i^{-1}(\Delta)) S_i^* \phi, S_i^* S_j E(\sigma_j^{-1}(\Delta)) S_j^* \psi \right\rangle \\ &= \left\langle E(\sigma_i^{-1}(\Delta)) S_i^* \phi, \delta_{i,j} id_{\mathcal{H}} E(\sigma_j^{-1}(\Delta)) S_j^* \psi \right\rangle \\ &= 0, \end{aligned}$$

for $i \neq j$. Thus, $\{P_i\}$ is a sequence of pairwise orthogonal projections on $\mathcal{H}$, and in consequence $\Phi(E)(\Delta)$ is well defined by Lemma 3. □

Next we recall a relationship between double and iterated limits of double sequences.

Let (Z, δ) be a metric space. We recall that a double sequence h_{lm} converges to $h \in Z$ and we write $\lim_{l,m\to\infty} h_{lm} = h$, if the following condition is satisfied: for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $\delta(h_{lm}, h) < \epsilon$ for all $l, m \geq N$. The element h is called the double limit of the double sequence $\{h_{lm}\}_{(l,m)\in\mathbb{N}^2}$.

It is clear to see that if the double sequence h_{lm} converges to h and $\lim_{m\to\infty} h_{l,m}$ exists for each l, then the iterated limit $\lim\limits_{l\to\infty} \lim\limits_{m\to\infty} h_{lm}$ exists and it is equal to h. We note that switching the roles of m and l yields the analogous result for the other iterated limit. So we have the following proposition.

Proposition 2 *Suppose that the double sequence* h_{lm} *converges to* $h \in Z$*. If* $\lim\limits_{l\to\infty} h_{lm}$ *exists for each* m*, and* $\lim\limits_{m\to\infty} h_{lm}$ *exists for each* l*, then the iterated limit* $\lim\limits_{m\to\infty}\lim\limits_{l\to\infty} h_{lm}$ *and* $\lim\limits_{l\to\infty}\lim\limits_{m\to\infty} h_{lm}$ *exist and both are equal to* h*.*

The following result is a consequence of the above lemma.

Lemma 5 *If* $\{P_{in}\}_{(i,n)\in\mathbb{N}^2}$ *is a sequence of pairwise orthogonal projections on* $\mathcal{H}$*, then* $\sum\limits_{i=1}^{\infty}\sum\limits_{n=1}^{\infty} P_{in}\phi = \sum\limits_{n=1}^{\infty}\sum\limits_{i=1}^{\infty} P_{in}\phi$ *for all* $\phi \in \mathcal{H}$*.*

Proof Consider a bijection $a : \mathbb{N} \to \mathbb{N}^2$. Since $\{P_{a(i)}\}_{i\in\mathbb{N}}$ is a sequence of pairwise orthogonal projections on $\mathcal{H}$, from Lemma 3, for each ϕ in $\mathcal{H}$, we have that

$$\sum_{i\in\mathbb{N}} P_{a(i)}\phi = P\phi, \tag{12}$$

where P is the orthogonal projection of $\mathcal{H}$ onto $\vee\{P_{a(i)}(\mathcal{H}) : \ i \geq 1\}$.
Let $\mathcal{F}$ be the collection of all finite subsets $F \subset \mathbb{N}^2$, and order $\mathcal{F}$ by inclusion, so $\mathcal{F}$ becomes a directed set. For each $F \in \mathcal{F}$ and $\phi \in \mathcal{H}$ fixed, define

$$P_F\phi = \sum_{j\in F} P_j\phi.$$

Then $\{P_F\phi : F \in \mathcal{F}\}$ is a net in $\mathcal{H}$. From [3, Definition 4.11, p.16] and (12) we have that $\{P_F\phi : F \in \mathcal{F}\}$ converge to Ph. Precisely, given a neighbourhood V of $P\phi$, there exists $F_0 \in \mathbb{N}^2$ such that $P_F\phi \in V$, for all $F \in \mathcal{F}$ with $F_0 \subset F$. In particular, given $\epsilon > 0$, consider the neighbourhood $V = \{\psi \in \mathcal{H} : \|\psi - P\phi\|_{\mathcal{H}} < \epsilon\}$. Then, there exists $F_0 = \{(x_1, y_1), (x_2, y_2), \ldots, (x_{n_0}, y_{n_0})\} \subset \mathbb{N}^2$ such that $P_F\phi \in V$, for all $F_0 \subset F$. Therefore, if

$$N = \max\{x_1, y_1, x_2, y_2, \ldots, x_{n_0}, y_{n_0}\}$$

and $F_{lm} = \{(i,k) \in \mathbb{N}^2 : 1 \leq i \leq l, 1 \leq k \leq m\}$, then

$$\left\| \sum_{(i,n)\in F_{lm}} P_{in}\phi - P\phi \right\|_{\mathcal{H}} < \epsilon, \text{ for all } l, m \geq N.$$

In consequence,

$$\lim_{l,m\to\infty} \sum_{i=1}^{l}\sum_{n=1}^{m} P_{in}\phi = P\phi.$$

Let $h_{lm} = \sum\limits_{i=1}^{l}\sum\limits_{n=1}^{m} P_{in}\phi$ for $(l,m) \in \mathbb{N}^2$. By Lemma 3, $\sum\limits_{i=1}^{\infty} P_{in}\phi$ is convergent for each $n \in \mathbb{N}$, and $\sum\limits_{n=1}^{\infty} P_{in}\phi$ is convergent for each $i \in \mathbb{N}$. Hence, $\lim\limits_{l\to\infty} h_{lm} =$

$\sum_{n=1}^{m}\sum_{i=1}^{\infty} P_{in}\phi$ for each m, and $\lim_{m\to\infty} h_{lm} = \sum_{i=1}^{l}\sum_{n=1}^{\infty} P_{in}\phi$ for each l. So, from Lemma 2 we conclude that

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} P_{in}\phi = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} P_{in}\phi = P\phi.$$

The proof is complete. □

Now, we will prove that the map Φ is well defined.

Theorem 5 *The map $\Phi : P(X) \to P(X)$ given by*

$$\Phi(E)(\cdot) = \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\cdot))S_i^*,$$

is well defined.

Proof Set $E \in P(X)$ and let $\phi, \psi \in \mathcal{H}$. By Lemma 4, $\Phi(E)(\Delta) \in B(\mathcal{H})$, for $\Delta \in \mathcal{B}(X)$. So, condition 1 of Definition 1 holds. Condition 2 of Definition 1 can be easily computed. Indeed,

$$\Phi(E)(\emptyset) = \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\emptyset))S_i^* = 0,$$

and according to Theorem 4 we get

$$\Phi(E)(X) = \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(X))S_i^* = \sum_{i=1}^{\infty} S_i S_i^* = id_{\mathcal{H}}.$$

To prove condition 3 of Definition 1, let $\Delta_1, \Delta_2 \in \mathcal{B}(X)$. We use the continuity of S_i^*, $i \in \mathbb{N}$, and Theorem 4 to obtain that

$$\begin{aligned}
\Phi(E)(\Delta_1)\Phi(E)(\Delta_2)\phi &= \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta_1))S_i^* \left(\sum_{j=1}^{\infty} S_j E(\sigma_j^{-1}(\Delta_2))S_j^*\phi\right) \\
&= \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta_1)) \left(\sum_{j=1}^{\infty} S_i^* S_j E(\sigma_j^{-1}(\Delta_2))S_j^*\phi\right) \\
&= \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta_1)) \left(\sum_{j=1}^{\infty} \delta_{i,j} E(\sigma_j^{-1}(\Delta_2))S_j^*\phi\right) \\
&= \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta_1))E(\sigma_i^{-1}(\Delta_2))S_i^*\phi \\
&= \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta_1) \cap \sigma_i^{-1}(\Delta_2))S_i^*\phi \\
&= \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta_1 \cap \Delta_2))S_i^*\phi \\
&= \Phi(E)(\Delta_1 \cap \Delta_2)\phi.
\end{aligned}$$

Finally, to prove condition 4 of Definition 1, let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{B}(X)$. As the pre-image preserves the union of sets, and $\sigma_i^{-1}(\Delta_n) \cap \sigma_i^{-1}(\Delta_m) = \emptyset$ for $n \neq m$, $i \in \mathbb{N}$, then the continuity of $\langle \cdot, \cdot \rangle$ leads to

$$\begin{aligned}
\left\langle \Phi(E)\left(\bigcup_{n=1}^{\infty} \Delta_n\right)\phi, \psi \right\rangle &= \left\langle \sum_{i=1}^{\infty} S_i E\left(\bigcup_{n=1}^{\infty} \sigma_i^{-1}(\Delta_n)\right) S_i^* \phi, \psi \right\rangle \\
&= \sum_{i=1}^{\infty} \left\langle S_i E\left(\bigcup_{n=1}^{\infty} \sigma_i^{-1}(\Delta_n)\right) S_i^* \phi, \psi \right\rangle \\
&= \sum_{i=1}^{\infty} \left\langle E\left(\bigcup_{n=1}^{\infty} \sigma_i^{-1}(\Delta_n)\right) S_i^* \phi, S_i^* \psi \right\rangle \\
&= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \left\langle E(\sigma_i^{-1}(\Delta_n)) S_i^* \phi, S_i^* \psi \right\rangle \\
&= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \left\langle S_i E(\sigma_i^{-1}(\Delta_n)) S_i^* \phi, \psi \right\rangle \\
&= \left\langle \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta_n)) S_i^* \phi, \psi \right\rangle.
\end{aligned}$$

Let $\{P_{in}\}_{(i,n)\in\mathbb{N}^2}$ be a sequence of bounded linear operator on $\mathcal{H}$ given by $P_{in} = S_i E(\sigma^{-1}(\Delta_n)) S_i^*$. A similar analysis that in the proof of Lemma 4 shows that P_{in} are orthogonal projections. In addition,

$$\begin{aligned}
\langle P_{in}\phi, P_{jm}\psi \rangle &= \left\langle E(\sigma_i^{-1}(\Delta_n)) S_i^* \phi, \delta_{i,j} id_{\mathcal{H}} E(\sigma_j^{-1}(\Delta_m)) S_j^* \psi \right\rangle \\
&= \delta_{i,j} \left\langle S_i^* \phi, E(\sigma_i^{-1}(\Delta_n)) E(\sigma_j^{-1}(\Delta_m)) S_j^* \psi \right\rangle \\
&= \delta_{i,j} \left\langle S_i^* \phi, E(\sigma_i^{-1}(\Delta_n) \cap \sigma_j^{-1}(\Delta_m)) S_j^* \psi \right\rangle \\
&= \delta_{i,j} \left\langle S_i^* \phi, \delta_{n,m} E(\sigma_i^{-1}(\Delta_n)) S_j^* \psi \right\rangle \\
&= 0,
\end{aligned}$$

for $(i,n) \neq (j,m)$. Thus, $\{P_{in}\}_{(i,n)\in\mathbb{N}^2}$ is a sequence of pairwise orthogonal projections on $\mathcal{H}$. Now, Lemma 5 shows that

$$\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta_n)) S_i^* \phi = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta_n)) S_i^* \phi.$$

So, by the continuity of $\langle \cdot, \cdot \rangle$ we obtain that

$$\begin{aligned}
\left\langle \Phi(E)\left(\bigcup_{n=1}^{\infty} \Delta_n\right)\phi, \psi \right\rangle &= \left\langle \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta_n)) S_i^* \phi, \psi \right\rangle \\
&= \left\langle \sum_{n=1}^{\infty} \Phi(E)(\Delta_n)\phi, \psi \right\rangle \\
&= \sum_{n=1}^{\infty} \left\langle \Phi(E)(\Delta_n)\phi, \psi \right\rangle.
\end{aligned}$$

The proof is complete. □

4 A projection-valued measure for a CIFS

In this section, we prove the main result of this paper, that is, the existence of a projection-valued measure for a CIFS.

First, we show that the map Φ given in (11) is a Lipschitz contraction on $(P(X),\rho)$. We start proving the following assertion:

$$\Phi(E)_{\phi,\phi}(\Delta)=\sum_{i=1}^{\infty}E_{S_i^*\phi,S_i^*\phi}(\sigma_i^{-1}(\Delta)), \tag{13}$$

for all $E\in P(X)$, $\Delta\in\mathcal{B}(X)$ and $\phi\in\mathcal{H}$. In fact,

$$\begin{aligned}\Phi(E)_{\phi,\phi}(\Delta)&=\langle\Phi(E)(\Delta)\phi,\phi\rangle=\sum_{i=1}^{\infty}\left\langle S_iE(\sigma_i^{-1}(\Delta))S_i^*\phi,\phi\right\rangle\\&=\sum_{i=1}^{\infty}\left\langle E(\sigma_i^{-1}(\Delta))S_i^*\phi,S_i^*\phi\right\rangle=\sum_{i=1}^{\infty}E_{S_i^*\phi,S_i^*\phi}(\sigma_i^{-1}(\Delta)).\end{aligned}$$

Theorem 6 *The map $\Phi:P(X)\to P(X)$ given by*

$$\Phi(E)(\cdot)=\sum_{i=1}^{\infty}S_iE(\sigma_i^{-1}(\cdot))S_i^*$$

is a Lipschitz contraction on $(P(X),\rho)$.

Proof Let $r=\sup_{i\in\mathbb{N}}\{r_i\}$ and $E,F\in P(X)$. Choose $f\in Lip_1(X)$. Since

$$\int f\ d\Phi(E)-\int f\ d\Phi(F)$$

is an operator self-adjoint,

$$\begin{aligned}&\left\|\int f\ d\Phi(E)-\int f\ d\Phi(F)\right\|\\&=\sup_{\|\phi\|_\infty=1}\left\{\left|\left\langle\left(\int f\ d\Phi(E)-\int f\ d\Phi(F)\right)\phi,\phi\right\rangle\right|\right\}.\end{aligned}$$

Let $\phi\in\mathcal{H}$ be with $\|\phi\|_\infty=1$. Then, by (13), Remark 1, Proposition 1 and Remark 3, we have that

$$\begin{aligned}&\left|\left\langle\left(\int f\ d\Phi(E)-\int f\ d\Phi(F)\right)\phi,\phi\right\rangle\right|\\&=\left|\left\langle\left(\int f\ d\Phi(E)\right)\phi,\phi\right\rangle-\left\langle\left(\int f\ d\Phi(F)\right)\phi,\phi\right\rangle\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_X f \; d\Phi(E)_{\phi,\phi} - \int_X f \; d\Phi(F)_{\phi,\phi} \right| \\
&= \left| \sum_{i=1}^{\infty} \int_X f \; d(E_{S_i^*\phi, S_i^*\phi} \circ \sigma_i^{-1})) - \sum_{i=1}^{\infty} \int_X f \; d(F_{S_i^*\phi, S_i^*\phi} \circ \sigma_i^{-1}) \right| \\
&= \left| \sum_{i=1}^{\infty} \int_X f \circ \sigma_i \; dE_{S_i^*\phi, S_i^*\phi} - \sum_{i=1}^{\infty} \int_X f \circ \sigma_i \; dF_{S_i^*\phi, S_i^*\phi} \right| \\
&= \left| \sum_{i=1}^{\infty} \left(\int_X f \circ \sigma_i \; dE_{S_i^*\phi, S_i^*\phi} - \int_X f \circ \sigma_i \; dF_{S_i^*\phi, S_i^*\phi} \right) \right| \\
&= r \left| \sum_{i=1}^{\infty} \left(\int_X \frac{f \circ \sigma_i}{r} \; dE_{S_i^*\phi, S_i^*\phi} - \int_X \frac{f \circ \sigma_i}{r} \; dF_{S_i^*\phi, S_i^*\phi} \right) \right| \\
&\leq r \sum_{i=1}^{\infty} \left| \int_X \frac{f \circ \sigma_i}{r} \; dE_{S_i^*\phi, S_i^*\phi} - \int_X \frac{f \circ \sigma_i}{r} \; dF_{S_i^*\phi, S_i^*\phi} \right| \\
&= r \sum_{i=1}^{\infty} \left| \left\langle \left(\int \frac{f \circ \sigma_i}{r} \; dE - \int \frac{f \circ \sigma_i}{r} \; dF \right) S_i^*\phi, S_i^*\phi \right\rangle \right| \\
&\leq r \sum_{i=1}^{\infty} \left(\left\| \int \frac{f \circ \sigma_i}{r} \; dE - \int \frac{f \circ \sigma_i}{r} \; dF \right\| \|S_i^*\phi\|_{\mathcal{H}}^2 \right).
\end{aligned}
$$

Note that the function $\frac{f \circ \sigma_i}{r} \in Lip_1(X)$, for all $i \in \mathbb{N}$. Hence, Theorem 4 implies that

$$
\begin{aligned}
&\left| \left\langle \left(\int f \; d\Phi(E) - \int f \; d\Phi(F) \right) \phi, \phi \right\rangle \right| \\
&\leq r\rho(E,F) \left(\sum_{i=1}^{\infty} \langle S_i^*\phi, S_i^*\phi \rangle \right) = r\rho(E,F) \left(\sum_{i=1}^{\infty} \langle S_i S_i^*\phi, \phi \rangle \right) \\
&= r\rho(E,F) \left\langle \left(\sum_{i=1}^{\infty} S_i S_i^* \right) \phi, \phi \right\rangle = r\rho(E,F) \langle \phi, \phi \rangle = r\rho(E,F),
\end{aligned}
$$

Therefore,

$$
\left\| \int f \; d\Phi(E) - \int f \; d\Phi(F) \right\| \leq r\rho(E,F).
$$

Since f is an arbitrary element of $Lip_1(X)$,

$$
\rho(\Phi(E), \Phi(F)) \leq r\rho(E,F),
$$

with $r < 1$. This proves that Φ is a Lipschitz contraction on $(P(X), \rho)$. □

By Theorems 1 and 6 we have that Φ is a contraction on the complete metric space $(P(X), \rho)$. So, by the Banach Fixed Point Theorem, we can deduce our main result, which extends [4, Theorem 1.7].

Theorem 7 *Let $\{\sigma_i : i \in \mathbb{N}\}$ be a CIFS on a compact metric space (Y, d) with Lipschitz constants r_i. Assume $r := \sup\{r_i : i \in \mathbb{N}\} < 1$ and let $X \subset Y$ be the attractor associated to the CIFS satisfying $\sigma_i(X) \cap \sigma_j(X) = \emptyset$ for $i \neq j$. We consider the isometries S_i and S_i^* defined in (10).Then there exists a unique projection-valued measure, $E \in P(X)$, such that*

$$E(\Delta) = \sum_{i=1}^{\infty} S_i E(\sigma_i^{-1}(\Delta)) S_i^*,$$

for all $\Delta \in \mathcal{B}(X)$.

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