Noname manuscript No.
(will be inserted by the editor)

On convergence of subspaces generated by horizontal dilations of polynomials. An application to best local approximation

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Received: date / Accepted: date

Abstract In this paper we study the convergence of a net of subspaces generated by horizontal dilations of polynomials in a finite dimensional subspace. As a consequence, we extend the results given by Zó and Cuenya [Proceedings of the Second International School. Advanced Courses of Mathematical Analysis II. (2007), 193-213] on a general approach to the problems of best vector-valued approximation on small regions from a finite dimensional subspace of polynomials.

Keywords Convergence of subspaces · Best local approximation · Abstract norms · Homogeneous dilations.

Mathematics Subject Classification (2010) 40A05 · 41A10 · 41A65

1 Introduction

Suppose that $\{a_j\}$ is a data set. This data are values of a function and its derivatives in a point. If we want to approximate these data using a polynomial of degree at most l, which will be the best algorithm to use? A Taylor polynomial of degree l is probably the most natural procedure to use.

This paper was partially supported by Universidad Nacional de Río Cuarto (grant PPI 18/C472), Universidad Nacional de San Luis (grant PROICO 22F623), and CONICET (grant PIP 112-201501-00433CO).

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The problem of finding an optimal algorithm to approximate a finite number of data corresponding to a function is developed in the best local approximation theory.

In 1934, Walsh proved in [9] that the Taylor polynomial of degree l for an analytic function f can be obtained by taking the limit as $\varepsilon \to 0$ of the best Chebyshev approximation to f from Π^l on the disk $|z| \leq \varepsilon$. This paper was the first association between the best local approximation to a function f from Π^l in 0 and the Taylor polynomial for f at the origin. However, the concept of best local approximation has been introduced and developed more recently by Chui, Shisha, and Smith in [1]. Later, several authors [2–8, 10] have studied this problem.

We consider a family of function seminorms $\{\|\cdot\|_\varepsilon\}_{\varepsilon>0}$, acting on Lebesgue measurable functions $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$, where B is the unit ball centered at the origin in $\mathbb{R}^n$. We will use the notation $F^\varepsilon(x) = F(\varepsilon x)$ and $\|F\|_\varepsilon^* = \|F^\varepsilon\|_\varepsilon$. For $l \in \mathbb{N} \cup \{0\}$, we will denote by Π^l the class of algebraic polynomials in n-variables of degree at most l, and Π_k^l the set $\{P = (p_1, \ldots, p_k) : p_s \in \Pi^l\}$.

Let $\mathcal{A}$ be a subspace of Π_k^l and let $\{P_\epsilon\}_{\varepsilon>0}$ be a net of best approximants to F from $\mathcal{A}$ respect to $\|\cdot\|_\varepsilon^*$, i.e.,

$$\|F - P_\varepsilon\|_\varepsilon^* \leq \|F - P\|_\varepsilon^*, \quad \text{for all} \quad P \in \mathcal{A}. \tag{1}$$

If the net $\{P_\epsilon\}_{\varepsilon>0}$ has a limit in $\mathcal{A}$ as $\epsilon \to 0$, this limit is called the *best local approximation to F from $\mathcal{A}$ in* 0. According to (1), we observe that $P_\varepsilon^\varepsilon$ is a polynomial in

$$\mathcal{A}^\varepsilon := \{P^\varepsilon : P \in \mathcal{A}\} \subset \Pi_k^l \tag{2}$$

of best approximation to F^ε by elements of the class $\mathcal{A}^\varepsilon$, respect to the seminorm $\|\cdot\|_\varepsilon$. We write it briefly by $P_\varepsilon^\varepsilon \in \mathcal{P}_{\mathcal{A}^\varepsilon,\varepsilon}(F^\varepsilon)$. Note that $\mathcal{A}^\varepsilon$ is a subspace generated by horizontal dilations the polynomials in $\mathcal{A}$.

From now on, we assume the following properties for the family of function seminorms $\|\cdot\|_\varepsilon$, $0 \leq \varepsilon \leq 1$.

(1) For $F = (f_1, \ldots, f_k)$ and $G = (g_1, \ldots, g_k)$, we have $\|F\|_\varepsilon \leq ||G||_\varepsilon$, for every $\varepsilon > 0$, whenever $|f_s| \leq |g_s|$, $s = 1, \ldots, k$.
(2) If 1 is the function $F(x) = (1, \ldots, 1)$, we have $\|1\|_\varepsilon < \infty$, for all $\varepsilon > 0$.
(3) For every $F \in C_k(B)$, we have $\|F\|_\varepsilon \to \|F\|_0$, as $\varepsilon \to 0$, where $C_k(B)$ is the set of continuous functions $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$. Moreover, $\|\cdot\|_0$ is a norm on $C_k(B)$.

An important point to note here is that there exist positive constants $C = C(m, k)$ and $\varepsilon(m, k)$ such that for every $0 < \varepsilon \leq \varepsilon(m, k)$,

$$\frac{1}{C}\|P\|_0 \leq \|P\|_\varepsilon \leq C\|P\|_0, \quad \text{for every} \quad P \in \Pi_k^m. \tag{3}$$

[11, Proposition 3.1]. For examples of nets of seminorms fulfilling conditions (1)-(3), we refer the reader to [11].

We say that $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$ has a Taylor polynomial of degree m at 0, if there exists $P \in \Pi_k^m$ such that

$$\|F - P\|_\varepsilon^* = o(\varepsilon^m), \quad \text{as} \quad \varepsilon \to 0.$$

It is well known that if it exists, it is unique and is denoted by $T_m = T_m(F)$ [11, Proposition 3.3]. We write $F \in t^m$ if the function F has the Taylor polynomial of degree m at 0. Moreover, if $F \in t^m$ and $T_m(F) = \sum_{|\alpha|\leq m} C_\alpha x^\alpha$, then the Taylor polynomial of degree $l \leq m$ for F at 0, is given by $T_l(F) = \sum_{|\alpha|\leq l} C_\alpha x^\alpha$ [11, Proposition 3.5]. We set $\partial^\alpha F(0)$ for the vector $\alpha! C_\alpha$.

The problem of best local approximation with a family of function seminorms $\{\|\cdot\|_\varepsilon\}_{\varepsilon>0}$ satisfying (1)-(3) was considered in [11] for two types of approximation class $\mathcal{A}$ fulfilling $\Pi_k^m \subset \mathcal{A} \subset \Pi_k^l$ and

(c1) $\mathcal{A}^\varepsilon = \mathcal{A}$, for each $\varepsilon > 0$, or
(c2) if $P \in \mathcal{A}$ and $T_{m+1}(P) = 0$, then $P = 0$.

Firstly, the authors studied the asymptotic behavior of a normalized error function as $\varepsilon \to 0$ [11, Theorems 4.2 and 4.5]. Secondly, they showed that there exists the best local approximation to F in 0 and is associated with a Taylor polynomial for F in 0 [11, Theorem 5.1]. In particular, if $\mathcal{A} = \Pi_k^m$ and $F \in t^m$, they proved that $P_\varepsilon \to T_m(F)$, as $\varepsilon \to 0$ [11, Theorem 3.1].

In this work we generalize the results found in [11], without the restrictions (c1) or (c2) given above. For this, it is essential to study the convergence of the net $\{\mathcal{A}^\varepsilon\}$ as $\varepsilon \to 0$.

This paper is organized as follows. In Section 2, we investigate the asymptotic behavior of $\{\mathcal{A}^\varepsilon\}$. In Section 3, we study the asymptotic behavior of the error function $\varepsilon^{-m-1}(F_\varepsilon - P_\varepsilon)^\varepsilon$ for a suitable integer, and we show some results about the best local approximation in the origin which generalizes those of [11].

2 Asymptotic behavior of the net $\{\mathcal{A}^\varepsilon\}$

In this section, we study the asymptotic behavior of the net $\{\mathcal{A}^\varepsilon\}$ given in (2). We begin with the following definition.

Definition 21 *Let $\mathcal{A} \subset \Pi_k^l$ be a subspace. We say that $P \in \lim_{\varepsilon\to 0} \mathcal{A}^\varepsilon$ if there exists a net $\{P_\epsilon\} \subset \mathcal{A}$ such that $\lim_{\varepsilon\to 0} \|P - P_\epsilon^\varepsilon\|_0 = 0$. We denote $\mathcal{B} = \lim_{\varepsilon\to 0} \mathcal{A}^\varepsilon$.*

Remark 22 *If $\mathcal{A} \subset \Pi_k^l$ is a subspace, then the sets $\mathcal{A}^\varepsilon$ and $\mathcal{B}$ are also subspaces of Π_k^l. Furthermore, if $\mathcal{A}^\varepsilon = \mathcal{A}$, for all $\varepsilon > 0$, we have that $\mathcal{B} = \mathcal{A}$.*

Proposition 23 *Let $\mathcal{A}$ be a subspace of polynomials such that $\Pi_k^m \subset \mathcal{A}$ for some $m \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. Then $\Pi_k^m \subset \mathcal{A}^\varepsilon$ for all $\varepsilon > 0$. Moreover, $\Pi_k^m \subset \mathcal{B}$.*

Proof Set $R_{\alpha,i}(x) = x^\alpha e_i$, $|\alpha| \leq m$, $1 \leq i \leq k$, where $\{e_i\}_{i=1}^k$ is the canonical basis of $\mathbb{R}^k$. Then

$$\{R_{\alpha,i} : |\alpha| \leq m, 1 \leq i \leq k\} \tag{4}$$

is a basis of the space Π_k^m. Since $\mathcal{A}^\varepsilon$ is a subspace, we have $R_{\alpha,i} = \frac{1}{\varepsilon^{|\alpha|}} R_{\alpha,i}^\varepsilon \in \mathcal{A}^\varepsilon$, and so $\Pi_k^m \subset \mathcal{A}^\varepsilon$, for all $\varepsilon > 0$. Finally, using the definition of $\mathcal{B}$, we obtain $\Pi_k^m \subset \mathcal{B}$.

From now on, for any Lebesgue measurable function $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$ we denote $T_{-1}(F) = 0$.

Proposition 24 *Let $\mathcal{A}$ be a subspace of Π_k^l and let $0 \leq s+1 \leq l$ be an integer. If $P \in \mathcal{A}$ satisfies $T_s(P) = 0$ and $T_{s+1}(P) \neq 0$, then $T_{s+1}(P) \in \mathcal{B}$.*

Proof For each $\varepsilon > 0$ we define $Q_\varepsilon = \frac{P}{\varepsilon^{s+1}} \in \mathcal{A}$. Since $T_s(P) = 0$, it follows that $\|T_{s+1}(P) - Q_\varepsilon^\varepsilon\|_0 = \frac{\|(T_{s+1}(P)-P)^\varepsilon\|_0}{\varepsilon^{s+1}}$. So $\|T_{s+1}(P) - Q_\varepsilon^\varepsilon\|_0 = o(1)$ as $\varepsilon \to 0$, and thus $T_{s+1}(P) \in \mathcal{B}$.

The following sets will be needed throughout the paper. Let $\mathcal{A}$ be a non-zero subspace of Π_k^l. We define

$$A_{-1} := \mathcal{A} \quad \text{and} \quad A_j := \{P \in \mathcal{A} : T_j(P) = 0\} \quad \text{for} \quad 0 \leq j \leq l. \tag{5}$$

We note that

$$A_j \subset A_i \quad \text{whenever} \quad i < j.$$

Since $A_l \subset \{P \in \Pi_k^l : T_l(P) = 0\} = \{0\}$, we have

$$\{j : 0 \leq j \leq l \text{ and } A_j \neq \mathcal{A}\} \neq \emptyset \quad \text{and} \quad \{j : 0 \leq j \leq l \text{ and } A_j = \{0\}\} \neq \emptyset.$$

Set

$$s_0 = \min\{j : 0 \leq j \leq l \text{ and } A_j \neq \mathcal{A}\}$$

and

$$r_0 = \min\{j : 0 \leq j \leq l \text{ and } A_j = \{0\}\}.$$

It easy to see that $0 \leq s_0 \leq r_0 \leq l$, and

$$s_0, r_0 \in \{j : s_0 \leq j \leq r_0 \text{ and } A_j \subsetneq A_{j-1}\} =: J. \tag{6}$$

We can now formulate our main result which describes the limit set $\mathcal{B}$.

Theorem 25 *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l. Then $\mathcal{B}$ is a subspace of $\Pi_k^{r_0}$ isomorphic to $\mathcal{A}$. Furthermore, under the above notation it is verified that*

(a) if $s_0 < r_0$ and $J \setminus \{r_0\} = \{s_0, \ldots, s_N\}$ with $s_i < s_{i+1}$ for $N > 0$, then $\mathcal{B} = T_{r_0}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \ldots \oplus T_{s_0}(S_{s_0})$, where $A_{s_i} \oplus S_{s_i} = A_{s_i-1}$, $0 \leq i \leq N$;
(b) if $s_0 = r_0$, then $\mathcal{B} = T_{r_0}(\mathcal{A})$.

Proof (*a*) Assume $s_0 < r_0$. Since every subspace of A_{s_i-1}, $0 \leq i \leq N$, has a complement, there exists a subspace $S_{s_i} \subset A_{s_i-1}$ such that

$$A_{s_i} \oplus S_{s_i} = A_{s_i-1}, \quad 0 \leq i \leq N. \tag{7}$$

In consequence,

$$\mathcal{A} = A_{s_N} \oplus S_{s_N} \oplus S_{s_{N-1}} \oplus \ldots \oplus S_{s_0}. \tag{8}$$

As $S_{s_i} \subset A_{s_i-1}$, $0 \leq i \leq N$, and $A_{r_0-1} = A_{s_N}$ we obtain

$$Q(x) = \begin{cases} \sum_{|\alpha| \geq s_i} \frac{\partial^\alpha Q(0)}{\alpha!} x^\alpha, & \text{if } Q \in S_{s_i}, \quad 0 \leq i \leq N. \\ \sum_{|\alpha| \geq s_{N+1}} \frac{\partial^\alpha Q(0)}{\alpha!} x^\alpha, & \text{if } Q \in A_{s_N}. \end{cases} \tag{9}$$

where $s_{N+1} = r_0$. Let $T_i : S_{s_i} \to \Pi_k^{s_i}$ be a linear operator defined by $T_i(P) = T_{s_i}(P)$, $0 \le i \le N$, and $T_{N+1} : \mathcal{A} \to \Pi_k^{s_{N+1}}$ be the linear operator given by $T_{N+1}(P) = T_{s_{N+1}}(P)$. We claim that
(i) T_i is an injective operator, $0 \le i \le N+1$.
(ii) $T_{s_{N+1}}(A_{s_N}) \cap \sum_{i=0}^{N} T_{s_i}(S_{s_i}) = \{0\}$.
(iii) If $N > 0$ then $T_{s_l}(S_{s_l}) \cap \left(T_{s_{N+1}}(A_{s_N}) + \sum_{i=0, i\neq l}^{N} T_{s_i}(S_{s_i})\right) = \{0\}$ whenever $l \neq i$.
Indeed, let $0 \le i \le N$. If $T_{s_i}(P) = T_{s_i}(Q)$ for some $P, Q \in S_{s_i}$, then $P - Q \in A_{s_i} \cap S_{s_i}$. So (7) implies that $P = Q$. On the other hand, if $T_{s_{N+1}}(P) = T_{s_{N+1}}(Q)$ with $P, Q \in \mathcal{A}$, then $P - Q \in A_{s_{N+1}} = \{0\}$, which proves (i). To prove (ii) we consider $Q_{N+1} \in A_{s_N}$ and $Q_i \in S_{s_i}$ such that $P = T_{s_{N+1}}(Q_{N+1}) = \sum_{i=0}^{N} T_{s_i}(Q_i)$. From (9) we see that

$$T_{s_{N+1}}(Q_{N+1})(x) = \sum_{|\alpha|=s_{N+1}} \frac{\partial^\alpha Q_N(0)}{\alpha!} x^\alpha \quad \text{and} \quad \sum_{i=0}^{N} T_{s_i}(Q_i) \in \Pi_k^{s_N}. \tag{10}$$

Therefore $P = 0$. Now, let $Q_{N+1} \in A_{s_N}$ and $Q_i \in S_{s_i}$ be such that

$$P = T_{s_l}(Q_l) = T_{s_{N+1}}(Q_{N+1}) + \sum_{i=0, i\neq l}^{N} T_{s_i}(Q_i). \tag{11}$$

From (9) it follows that

$$T_{s_i}(Q_i) = \sum_{|\alpha|=s_i} \frac{\partial^\alpha Q_i(0)}{\alpha!} x^\alpha, \quad 0 \le i \le N.$$

According to (10) and (11) we have $P = 0$, and (iii) is proved.
Using (i)-(iii), we deduce that the subspace

$$T_{s_{N+1}}(A_{s_N}) + T_{s_N}(S_{s_N}) + T_{s_{N-1}}(S_{s_{N-1}}) + \ldots + T_{s_0}(S_{s_0})$$

is a direct sum isomorphic to $\mathcal{A}$. The proof concludes by proving

$$\mathcal{B} = T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \ldots \oplus T_{s_0}(S_{s_0}). \tag{12}$$

We observe that if $P \in S_{s_i} \setminus \{0\}$, then $T_{s_i}(P) \neq 0$ and $T_{s_i-1}(P) = 0$ by (7). So, Proposition 24 implies that $T_{s_i}(P) \in \mathcal{B}$. On the other hand, if $P \in A_{s_N} \setminus \{0\}$, we get $T_{s_N}(P) = 0$. Moreover, we have $T_{s_{N+1}}(P) \neq 0$. In fact, on the contrary, we see that $P \in A_{s_{N+1}} = \{0\}$. Proposition 24 now gives $T_{s_{N+1}}(P) \in \mathcal{B}$. Therefore,

$$T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \ldots \oplus T_{s_0}(S_{s_0}) \subset \mathcal{B}.$$

On the other hand, if $P \in \mathcal{B}$, there exists $\{P_\epsilon\} \subset \mathcal{A}$ such that

$$\lim_{\varepsilon \to 0} ||P - P_\epsilon^\varepsilon||_0 = 0. \tag{13}$$

Let d_{N+1} =dim(A_{s_N}) and d_i =dim(S_{s_i}), $0 \le i \le N$. We take $\{v_l\}_{l=1}^{d_{N+1}}$ and $\{w_{ir}\}_{r=1}^{d_i}$ basis of A_{s_N}, and S_{s_i} respectively. It is easy to check that for each

$0 < \varepsilon \leq 1$, $\{\varepsilon^{-s_{N+1}} v_l\}_{l=1}^{a}$ is a basis of A_{s_N} and $\{\varepsilon^{-s_i} w_{ir}\}_{r=1}^{d_i}$ is a basis of S_{s_i}, $0 \leq i \leq N$. According to (8), we have that there exist real numbers. $D_{l,\varepsilon}, C_{i,r,\varepsilon}$ such that

$$P_\varepsilon = \sum_{l=1}^{d_{N+1}} \varepsilon^{-s_{N+1}} D_{l,\varepsilon} v_l + \sum_{i=0}^{N} \sum_{r=1}^{d_i} \varepsilon^{-s_i} C_{i,r,\varepsilon} w_{ir}.$$

From (9) it follows that

$$v_l(x) = \sum_{|\alpha| \geq s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha \quad \text{and} \quad w_{ir}(x) = \sum_{|\alpha| \geq s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha.$$

Consequently,

$$\begin{aligned}
P_\varepsilon^\varepsilon(x) &= \sum_{l=1}^{d_{N+1}} D_{l,\varepsilon} \varepsilon^{-s_{N+1}} v_l^\varepsilon(x) + \sum_{i=0}^{N} \sum_{r=1}^{d_i} C_{i,r,\varepsilon} \varepsilon^{-s_i} w_{ir}^\varepsilon(x) \\
&= \sum_{l=1}^{d_{N+1}} \sum_{|\alpha| = s_{N+1}} D_{l,\varepsilon} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha + \sum_{l=1}^{d_{N+1}} \sum_{|\alpha| > s_{N+1}} D_{l,\varepsilon} \varepsilon^{|\alpha| - s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha \\
&+ \sum_{i=0}^{N} \sum_{r=1}^{d_i} \sum_{|\alpha| = s_i} C_{i,r,\varepsilon} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha + \sum_{i=0}^{N} \sum_{r=1}^{d_i} \sum_{|\alpha| > s_i} C_{i,r,\varepsilon} \varepsilon^{|\alpha| - s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha \\
&= \sum_{l=1}^{d_{N+1}} D_{l,\varepsilon} T_{s_{N+1}}(v_l)(x) + \sum_{i=0}^{N} \left(\sum_{r=1}^{d_i} C_{i,r,\varepsilon} T_{s_i}(w_{ir})(x) \right) \\
&+ \sum_{l=1}^{d_{N+1}} \sum_{|\alpha| > s_{N+1}} D_{l,\varepsilon} \varepsilon^{|\alpha| - s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha \\
&+ \sum_{i=0}^{N} \sum_{r=1}^{d_i} \sum_{|\alpha| > s_i} C_{i,r,\varepsilon} \varepsilon^{|\alpha| - s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha.
\end{aligned}$$

An straightforward computation shows that

$$\begin{aligned}
T_{s_0}(P_\varepsilon^\varepsilon)(x) &= \sum_{r=1}^{d_0} C_{0,r,\varepsilon} T_{s_0}(w_{0r})(x) \\
T_{s_j}(P_\varepsilon^\varepsilon)(x) &= T_{s_{j-1}}(P_\varepsilon^\varepsilon)(x) + \sum_{i=0}^{j-1} \sum_{r=1}^{d_i} \sum_{s_i < |\alpha| \leq s_j} C_{i,r,\varepsilon} \varepsilon^{|\alpha| - s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha \\
&+ \sum_{r=1}^{d_j} C_{j,r,\varepsilon} T_{s_j}(w_{jr})(x),
\end{aligned}$$

$1 \le j \le N$, and

$$T_{s_{N+1}}(P_\varepsilon^\varepsilon)(x) = T_{s_N}(P_\varepsilon^\varepsilon)(x) + \sum_{i=0}^{N}\sum_{r=1}^{d_i}\sum_{s_i<|\alpha|\le s_{N+1}} C_{i,r,\varepsilon}\varepsilon^{|\alpha|-s_i}\frac{\partial^\alpha w_{ir}(0)}{\alpha!}x^\alpha$$
$$+ \sum_{l=1}^{d_{N+1}} D_{l,\varepsilon}T_{s_{N+1}}(v_l)(x).$$

Since $\{T_{s_{N+1}}(v_l)\}_{l=1}^a$ is a basis of $T_{s_{N+1}}(A_{s_N})$ and $\{T_i(w_{ir})\}_{r=1}^{d_i}$ is a basis of $T_i(S_{s_i})$, $0 \le i \le N$, (13) shows that there are real numbers. D_l and $C_{i,r}$ such that $D_{l,\varepsilon}, \to D_l$ and $C_{i,r,\varepsilon} \to C_{i,r}$, as $\varepsilon \to 0$. In consequence,

$$P = \sum_{l=1}^{a} D_l T_{s_{N+1}}(v_l) + \sum_{i=0}^{N}\left(\sum_{r=1}^{d_i} C_{i,r}T_{s_i}(w_{ir})\right),$$

and so $P \in T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \ldots \oplus T_{s_0}(S_{s_0})$.
(b) Now assume $s_0 = r_0$, i.e. $A_{s_0} = \{0\}$. Then $\mathcal{A}$ has the form (8) with $N = 0$, $A_{s_0} = \{0\}$ and $S_{s_0} = \mathcal{A}$. An analysis similar to the proof of (a) shows that T_{r_0} is an isomorphism and $\mathcal{B} = T_{s_0}(S_{s_0}) = T_{r_0}(\mathcal{A})$.

The following corollary follows immediately from the proof of Theorem 25.

Corollary 26 *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l. Then $\lim_{n\to\infty} \mathcal{A}^{\varepsilon_n} = \mathcal{B}$ for any sequence $\{\varepsilon_n\}$ of the net $\epsilon \downarrow 0$.*

Remark 27 *$\mathcal{B}$ is isomorphic to $T_{r_0}(\mathcal{A})$.*

Corollary 28 *Let $s \ge m+1$ and let $\mathcal{A} = \Pi_k^m \oplus A_{s-1}$ be such that $A_s = \{0\}$. Then $\mathcal{B} = \Pi_k^m \oplus T_s(A_{s-1})$ and the linear operator $T : \mathcal{A} \to \Pi_k^s$ given by $T(P) = T_s(P)$ define an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.*

Proof We first claim that T is an injective operator. Indeed, if $T(P) = T(Q)$ for $P, Q \in \mathcal{A}$, then $T_s(P-Q) = 0$ and so $P - Q \in A_s$. Since $A_s = \{0\}$, we have $P = Q$.
As $\mathcal{A}$ is isomorphic to $T(\mathcal{A})$, the proof concludes by proving $\mathcal{B} = \Pi_k^m \oplus T_s(A_{s-1}) = T_s(\mathcal{A})$.
Let A_j be the sets defined in (5). Since

$$\{0\} = A_s \subsetneq A_{s-1} = \ldots = A_m \subsetneq A_{m-1} \subsetneq \ldots \subsetneq A_0 \subsetneq \mathcal{A},$$

then $\mathcal{A} = A_{s-1} \oplus B_m \oplus B_{m-1} \oplus \ldots \oplus B_0$, where $A_i \oplus B_i = A_{i-1}$, $0 \le i \le m$. Therefore Π_k^m is isomorphic to $B_m \oplus \ldots \oplus B_0$. On the other hand, since $s_0 = 0$, $r_0 = s$ and $J \setminus \{r_0\} = \{0, 1, \ldots, m\}$, by Proposition 25 (a),

$$\mathcal{B} = T_s(A_{s-1}) \oplus T_m(B_m) \oplus \ldots \oplus T_0(B_0).$$

From the proof of Theorem 25, we obtain that $B_m \oplus \ldots \oplus B_0$ is isomorphic to $T_m(B_m) \oplus \ldots \oplus T_0(B_0)$, and consequently Π_k^m is isomorphic to $T_m(B_m) \oplus \ldots \oplus T_0(B_0) \subset \Pi_k^m$. Hence, $T_m(B_m) \oplus \ldots \oplus T_0(B_0) = \Pi_k^m$ and so $\mathcal{B} = T_s(A_{s-1}) \oplus \Pi_k^m = T_s(A_{s-1}) \oplus T_s(\Pi_k^m) = T_s(\mathcal{A})$.

3 An application to best local approximation

Let $\{P_\epsilon\}$ be a net of best approximants to F from $\mathcal{A}$ respect to $\|\cdot\|_\varepsilon^*$, and let E_ε be the error function

$$E_\varepsilon(F) = \frac{F^\varepsilon - P_\varepsilon^\varepsilon}{\varepsilon^{m+1}}.$$

If $F \in t^{m+1}$, then

$$F^\varepsilon = T_{m+1}^\varepsilon + \varepsilon^{m+1} R_{m+1}^\varepsilon \quad \text{where} \quad R_{m+1} = \frac{F - T_{m+1}}{\varepsilon^{m+1}}, \quad ||R_{m+1}^\varepsilon||_\varepsilon = o(1),$$

and T_{m+1} is the Taylor polynomial of F of degree $m+1$ at 0. Moreover,

$$\lambda P_\varepsilon^\varepsilon \in \mathcal{P}_{\mathcal{A}^\varepsilon,\varepsilon}(\lambda F^\varepsilon) \quad \text{and} \quad P^\varepsilon + P_\varepsilon^\varepsilon \in \mathcal{P}_{\mathcal{A}^\varepsilon,\varepsilon}((P+F)^\varepsilon), \quad \text{for} \quad P \in \mathcal{A}.$$

The following proposition may be proved in much the same way as [11, Proposition 4.1]. However, we repeat the proof by completeness.

Proposition 31 *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l with $l > m$, and let $\{P_\epsilon\}$ be a net of best approximants of F from $\mathcal{A}$ respect to $\|\cdot\|_\varepsilon^*$. If $F \in t^{m+1}$, $T_m \in \mathcal{A}$ and $\phi_{m+1} = T_{m+1} - T_m$, then*

$$E_\varepsilon(F) = \phi_{m+1} + R_{m+1}^\varepsilon - \mathcal{P}_{\mathcal{A}^\varepsilon,\varepsilon}(\phi_{m+1} + R_{m+1}^\varepsilon),$$

where $\|R_{m+1}^\varepsilon\|_\varepsilon = o(1)$, as $\varepsilon \to 0$.

Proof Since $R_{m+1}^\varepsilon = \frac{F^\varepsilon - T_{m+1}^\varepsilon}{\varepsilon^{m+1}}$, then

$$\begin{aligned}\phi_{m+1} + R_{m+1}^\varepsilon &= T_{m+1} - T_m + \frac{F^\varepsilon - T_{m+1}^\varepsilon}{\varepsilon^{m+1}} = \frac{T_{m+1}^\varepsilon - T_m^\varepsilon}{\varepsilon^{m+1}} + \frac{F^\varepsilon - T_{m+1}^\varepsilon}{\varepsilon^{m+1}} \\ &= \frac{F^\varepsilon - T_m^\varepsilon}{\varepsilon^{m+1}}.\end{aligned}$$

As $T_m \in \mathcal{A}$, we have

$$\begin{aligned}\phi_{m+1} + R_{m+1}^\varepsilon - \mathcal{P}_{\mathcal{A}^\varepsilon,\varepsilon}(\phi_{m+1} + R_{m+1}^\varepsilon) &= \frac{F^\varepsilon - T_m^\varepsilon}{\varepsilon^{m+1}} - P_{\mathcal{A}^\varepsilon,\varepsilon}\left(\frac{F^\varepsilon - T_m^\varepsilon}{\varepsilon^{m+1}}\right) \\ &= \frac{F^\varepsilon - P_\varepsilon^\varepsilon}{\varepsilon^{m+1}} = E_\varepsilon(F).\end{aligned}$$

Next, we give a new result about the asymptotic behavior of error without the conditions (c1) or (c2), which generalizes Theorems 4.2 and 4.5 given in [11].

Theorem 32 *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l with $l > m$. If $F \in t^{m+1}$, $T_m \in \mathcal{A}$ and $\phi_{m+1} = T_{m+1} - T_m$, then*

$$\|E_\varepsilon(F)\|_\varepsilon \to \inf_{P \in \mathcal{B}} \|\phi_{m+1} - P\|_0, \quad \text{as} \quad \varepsilon \to 0.$$

Proof By Proposition 31,

$$E_\varepsilon(F) = \phi_{m+1} + R^\varepsilon_{m+1} - \mathcal{P}_{\mathcal{A}^\varepsilon,\varepsilon}(\phi_{m+1} + R^\varepsilon_{m+1}), \tag{14}$$

where $\|R^\varepsilon_{m+1}\|_\varepsilon = o(1)$ as $\varepsilon \to 0$. We first prove

$$\overline{\lim_{\varepsilon\to 0}} ||E_\varepsilon(F)||_\varepsilon \le \inf_{P\in B} \|\phi_{m+1} - P\|_0. \tag{15}$$

In fact, let $P \in \mathcal{B}$. By the definition of $\mathcal{B}$, there exists a net $\{Q_\varepsilon\} \subset \mathcal{A}$ such that $\|P - Q^\varepsilon_\varepsilon\|_0 \to 0$, as $\varepsilon \to 0$. In consequence, $\|P - Q^\varepsilon_\varepsilon\|_\varepsilon = o(1)$, as $\varepsilon \to 0$, by (3). Since $Q^\varepsilon_\varepsilon \in \mathcal{A}^\varepsilon$ and $\|R^\varepsilon_{m+1}\|_\varepsilon = o(1)$, from (14) we obtain

$$\|E_\varepsilon(F)\|_\varepsilon \le \|\phi_{m+1}+R^\varepsilon_{m+1}-Q^\varepsilon_\varepsilon\|_\varepsilon \le \|\phi_{m+1}-Q^\varepsilon_\varepsilon\|_\varepsilon+o(1), \quad \text{as} \quad \varepsilon \to 0. \tag{16}$$

By Property (3), $\|\phi_{m+1}-P\|_\varepsilon \to \|\phi_{m+1}-P\|_0$, as $\varepsilon \to 0$. Hence, using Triangle Inequality we have

$$\begin{aligned} |\|\phi_{m+1} - Q^\varepsilon_\varepsilon\|_\varepsilon - \|\phi_{m+1} - P\|_0| &\le |\|\phi_{m+1} - Q^\varepsilon_\varepsilon\|_\varepsilon - \|\phi_{m+1} - P\|_\varepsilon| \\ &\quad + |\|\phi_{m+1} - P\|_\varepsilon - \|\phi_{m+1} - P\|_0| \\ &\le \|P - Q^\varepsilon_\varepsilon\|_\varepsilon + |\|\phi_{m+1} - P\|_\varepsilon - \|\phi_{m+1} - P\|_0| = o(1). \end{aligned}$$

as $\varepsilon \to 0$. Now, according to (16) we get (15).
The proof finishes by observing that

$$\underline{\lim_{\varepsilon\to 0}} \|E_\varepsilon(F)\|_\varepsilon \ge \inf_{P\in\mathcal{B}} \|\phi_{m+1} - P\|_0. \tag{17}$$

Let $\varepsilon \downarrow 0$ be a sequence such that $\lim_{\varepsilon\to 0} \|E_\varepsilon(F)\|_\varepsilon = \underline{\lim}_{\varepsilon\to 0} \|E_\varepsilon(F)\|_\varepsilon$. We consider $P^\varepsilon_\varepsilon \in \mathcal{P}_{\mathcal{A}^\varepsilon,\varepsilon}(\phi_{m+1}+R^\varepsilon_{m+1})$. We claim that there exist constants $M, \varepsilon_0 > 0$ such that

$$\|P^\varepsilon_\varepsilon\|_0 \le M, \quad 0 < \varepsilon \le \varepsilon_0. \tag{18}$$

Indeed, as $0 \in \mathcal{A}^\varepsilon$ we get

$$\begin{aligned} \|P^\varepsilon_\varepsilon\|_\varepsilon &\le \|P^\varepsilon_\varepsilon - (\phi_{m+1} + R^\varepsilon_{m+1})\|_\varepsilon + \|\phi_{m+1} + R^\varepsilon_{m+1}\|_\varepsilon \\ &\le 2\|\phi_{m+1} + R^{\varepsilon_n}_{m+1}\|_\varepsilon \\ &\le 2\|\phi_{m+1}\|_\varepsilon + 2\|R^\varepsilon_{m+1}\|_\varepsilon, \end{aligned} \tag{19}$$

for $0 < \varepsilon \le 1$. By Proposition 31 and Property (3), we see that $2\|\phi_{m+1}\|_\varepsilon + 2\|R^\varepsilon_{m+1}\|_\varepsilon \to 2\|\phi_{m+1}\|_0$, as $\varepsilon \to 0$. So, from (3) and (19), we obtain (18).
In consequence, there exists a subsequence of $\{P^\varepsilon_\varepsilon\}$, which is denoted in the same way, and $P_0 \in \Pi^l_k$ such that $P^\varepsilon_\varepsilon \to P$ uniformly on B, as $\varepsilon \to 0$. Since $|\|\phi_{m+1} - P^\varepsilon_\varepsilon\|_\varepsilon - \|\phi_{m+1} - P\|_0| \le |\|\phi_{m+1} - P^\varepsilon_\varepsilon\|_\varepsilon - \|\phi_{m+1} - P\|_\varepsilon| + |\|\phi_{m+1} - P\|_\varepsilon - \|\phi_{m+1} - P\|_0| \le \|P - P^\varepsilon_\varepsilon\|_\varepsilon + |\|\phi_{m+1} - P\|_\varepsilon - \|\phi_{m+1} - P\|_0|$, using Property (3) we get

$$\|\phi_{m+1} - P\|_0 = \|\phi_{m+1} - P^\varepsilon_\varepsilon\|_\varepsilon + o(1), \quad \text{as} \quad \varepsilon \to 0.$$

We observe that $P \in B$ by Corolary 26. Therefore, by Proposition 31,

$$\inf_{Q\in\mathcal{B}} \|\phi_{m+1} - Q\|_0 \le \|\phi_{m+1} - P\|_0 = \|\phi_{m+1} - P_\varepsilon^\varepsilon\|_\varepsilon + o(1)$$
$$\le \|\phi_{m+1} + R_{m+1}^\varepsilon - P_\varepsilon^\varepsilon\|_\varepsilon + \|R_{m+1}^\varepsilon\|_\varepsilon$$
$$= \|E_\varepsilon(F)\|_\varepsilon + \|R_{m+1}^\varepsilon\|_\varepsilon.$$

So, $\inf_{Q\in\mathcal{B}} \|\phi_{m+1} - Q\|_0 \le \lim_{\varepsilon\to 0} \left(\|E_\varepsilon(F)\|_\varepsilon + \|R_{m+1}^\varepsilon\|_\varepsilon\right) = \underline{\lim}_{\varepsilon\to 0} \|E_\varepsilon(F)\|_\varepsilon$, and (17) is proved.

The following result provides us with a useful and important property for a net of best approximants to F from $\mathcal{A}$.

Theorem 33 *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l with $l > m$, and let $\{P_\epsilon\}$ be a net of best approximants of F from $\mathcal{A}$ respect to $\|\cdot\|_\varepsilon^*$. Assume $F \in t^{m+1}$, $T_m \in \mathcal{A}$ and $\phi_{m+1} = T_{m+1} - T_m$. If $\mathcal{C}$ is the cluster point set of the net $\left\{\frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}}\right\}$, as $\varepsilon \to 0$, then $\mathcal{C} \neq \emptyset$. Moreover, each polynomial in $\mathcal{C}$ is a solution of the minimization problem:*

$$\min_{P\in\mathcal{B}} \|\phi_{m+1} - P\|_0. \tag{20}$$

Proof We observe

$$E_\varepsilon(F) = \frac{(F - P_\varepsilon)^\varepsilon}{\varepsilon^{m+1}} = \frac{(T_{m+1} - T_m)^\varepsilon + (F - T_{m+1})^\varepsilon - (P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}}$$
$$= \frac{\phi_{m+1}^\varepsilon - (P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}} + \frac{(F - T_{m+1})^\varepsilon}{\varepsilon^{m+1}}$$
$$= \phi_{m+1} - \frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}} + \frac{(F - T_{m+1})^\varepsilon}{\varepsilon^{m+1}}.$$

Then

$$\left\|\phi_{m+1} - \frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}}\right\|_\varepsilon - \frac{||(F - T_{m+1})^\varepsilon||_\varepsilon}{\varepsilon^{m+1}} \le ||E_\varepsilon(F)||_\varepsilon$$
$$\le \left\|\phi_{m+1} - \frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}}\right\|_\varepsilon + \frac{||(F - T_{m+1})^\varepsilon||_\varepsilon}{\varepsilon^{m+1}},$$

and consequently,

$$||E_\varepsilon(F)||_\varepsilon = \left\|\phi_{m+1} - \frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}}\right\|_\varepsilon + o(1), \quad \text{as} \quad \varepsilon \to 0,$$

since $F \in t^{m+1}$. By Theorem 32,

$$\inf_{P\in B} ||\phi_{m+1} - P||_0 = \lim_{\varepsilon\to 0} \left\|\phi_{m+1} - \frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}}\right\|_\varepsilon. \tag{21}$$

According to (3), there exist constants $\varepsilon_0, M > 0$ such that

$$\left\|\phi_{m+1} - \frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}}\right\|_0 \le M,$$

for all $0 < \varepsilon \leq \varepsilon_0$. The equivalence of the norms in Π_k^l implies that the net $\left\{\frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}}\right\}_{0<\varepsilon\leq\varepsilon_0}$ is uniformly bounded on B. So, there exists a subsequence of $\left\{\frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}}\right\}_{0<\varepsilon\leq\varepsilon_0}$, which is denoted in the same way, and a polynomial P_0 such that

$$\frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}} \text{ converge a } P_0, \text{ uniformly on } B, \text{ as } \varepsilon \to 0. \tag{22}$$

In consequence, $\mathcal{C} \neq \emptyset$.
On the other hand, if $P_0 \in \mathcal{C}$, there is a sequence $\varepsilon \downarrow 0$ such that $\frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}} \to P_0$. Since $T_m \in \mathcal{A}$, we have $P_\varepsilon - T_m \in \mathcal{A}$, and so $P_0 \in \mathcal{B}$ by Corollary 26. Finally, from Property (3) and (21) we conclude that

$$\inf_{P\in B} ||\phi_{m+1} - P||_0 = \lim_{\varepsilon\to 0} \left\| \phi_{m+1} - \frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}} \right\|_\varepsilon = \|\phi_{m+1} - P_0\|_0 ,$$

i.e. P_0 is a solution of (20).

The following theorem is an extension of [11, Theorem 5.1].

Theorem 34 *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l with $l > m$, and let $\{P_\epsilon\}$ be a net of best approximants of F from $\mathcal{A}$ respect to $\|\cdot\|_\varepsilon^*$. Assume $m+1 = \min\{j : 0 \leq j \leq l \text{ and } A_j = \{0\}\}$, $F \in t^{m+1}$ with $T_m \in \mathcal{A}$ and set $\phi_{m+1} = T_{m+1} - T_m$. If the minimization problem (20) has a unique solution P_0, then $P_\varepsilon \to T_m + P$, where $P \in \mathcal{A}$ is uniquely determined by the condition $T_{m+1}(P) = P_0 - T_m(P_0)$.*

Proof Since (20) has a unique solution P_0, Theorem 33 implies that

$$\lim_{\varepsilon\to 0} \frac{(P_\varepsilon - T_m)^\varepsilon}{\varepsilon^{m+1}} = P_0.$$

In consequence, $\partial^\alpha(P_\varepsilon - T_m)(0) \to 0$, $|\alpha| \leq m$, and $\partial^\alpha(P_\varepsilon - T_m)(0) \to \partial^\alpha P_0(0)$, $|\alpha| = m+1$, as $\varepsilon \to 0$. Therefore

$$T_{m+1}(P_\epsilon - T_m)(x) \to \sum_{|\alpha|=m+1} \frac{\partial^\alpha P_0(0)}{\alpha!} x^\alpha =: R(x), \; x \in B, \text{ as } \varepsilon \to 0. \tag{23}$$

Let $T : \mathcal{A} \to \Pi_k^{m+1}$ be the linear operator defined by $T(P) = T_{m+1}(P)$. As $A_{m+1} = \{0\}$, an analysis similar to that in the proof of Corollary 28 shows that T is an injective operator. Since $T(\mathcal{A})$ is a closed subspace and $\{T_{m+1}(P_\epsilon - T_m)\} \subset T(\mathcal{A})$, (23) implies that there exists a unique $P \in \mathcal{A}$ such that $T_{m+1}(P) = R$. Hence $T_{m+1}(P_\varepsilon - T_m - P) \to 0$ as $\varepsilon \to 0$. As $A_{m+1} = \{0\}$ we see that $\|Q\| := \|T_{m+1}(Q)\|_0$ is a norm on $\mathcal{A}$, and so $P_\varepsilon \to T_m + P$ as $\varepsilon \to 0$. Finally, by Theorem 25, $\mathcal{B} \subset \Pi_k^{m+1}$, and consequently $P_0 - T_m(P_0) = T_{m+1}(P_0) - T_m(P_0) = R$. The proof is complete.

Remark 35 *If $\mathcal{A}$ satisfies the condition (c2), then $\mathcal{A} = \Pi_k^m \oplus A_m$ with $A_{m+1} = \{0\}$. By Corollary 28, $\mathcal{B} = \Pi_k^m \oplus T_{m+1}(A_m)$ and each element $P \in \mathcal{A}$ is uniquely determined by $T_{m+1}(P)$. So, we can rewrite the problem (20) in the following (equivalent) form:*

$$\min_{Q+U\in \Pi_k^m \oplus A_m} \|\phi_{m+1} - (Q + T_{m+1}(U))\|_0. \tag{24}$$

The following result has been proved in [11, Theorem 5.1] and it is a consequence of Theorem 34.

Corollary 36 *Let $\Pi_k^m \subset \mathcal{A} \subset \Pi_k^l$ be a non-zero subspace that satisfies the condition (c2) and let $\{P_\epsilon\}$ be a net of best approximants of F from $\mathcal{A}$ respect to $\|\cdot\|_\varepsilon^*$. Assume $F \in t^{m+1}$. If the minimization problem (24) has a unique solution P_0, then $P_\varepsilon \to T_m + P$, where $P \in \mathcal{A}$ is uniquely determined by the condition $T_{m+1}(P) = P_0 - T_m(P_0)$.*

In the following example we present a function $F \in \bigcap_{m=0}^{\infty} t^m$ such that $T_2(F) \notin \mathcal{A}$ and the net $\{T_i(P_\varepsilon)\}$ does not converge for the same $i > m + 1$.

Example 37 *Set $B = [-1, 1]$, $\|G\|_\varepsilon = \left(\int_{-1}^{1} |G(x)|^2 dx\right)^{\frac{1}{2}}$, $F(x) = x$, and $\mathcal{A} = span\{1, x^2, x^3\}$. So*

$$\|G\|_\varepsilon^* = \left(\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} |G(x)|^2 dx\right)^{\frac{1}{2}},$$

$A_0 = A_1 = span\{x^2, x^3\}$, $A_2 = span\{x^3\}$ and $A_3 = \{0\}$. Since $T_1(x^2) = 0$, we observe that the subspace $\mathcal{A}$ does not satisfies the condition (c2). Moreover, an straightforward computation shows that

$$\frac{\|F - T_0\|_\varepsilon^*}{\varepsilon^0} = \frac{\sqrt{6}}{3}\varepsilon \quad \text{and} \quad \frac{\|F - T_s\|_\varepsilon^*}{\varepsilon^s} = 0, \quad s \in \mathbb{N},$$

where $T_0(x) = 0$ and $T_s(x) = x$. In consequence, $F \in t^m$ for all $m \in \mathbb{N} \cup \{0\}$, and $T_2(F) \notin \mathcal{A}$. Since $\int_{-\varepsilon}^{\varepsilon} \left(x - \frac{7}{5\varepsilon^2}x^3\right) x^i dx = 0$, $i = 0, 2, 3$, then $P_\varepsilon(x) = \frac{7}{5\varepsilon^2}x^3$ is the best approximant to F from $\mathcal{A}$ respect to $\|\cdot\|_\varepsilon^*$. Therefore $T_i(P_\varepsilon)(x) \to 0$, for $i = 0, 1, 2$, but $T_3(P_\varepsilon)(x)$ does not converge, as $\varepsilon \to 0$. So, the best local approximation to F from $\mathcal{A}$ in 0 does not exist, and

$$\|E_\varepsilon(F)\|_\varepsilon = \frac{\|F - P_\varepsilon\|_\varepsilon^*}{\varepsilon^3} = \frac{2\sqrt{6}}{15\varepsilon^2} \to \infty, \quad as \quad \varepsilon \to 0.$$

We now give another example which shows that the condition $T_m \in \mathcal{A}$ is not necessary for the existence of the best local approximation.

Example 38 *Set B, $\|\cdot\|_\varepsilon^*$ and F as in Example 37, and we consider the subspace $\mathcal{A} = span\{1, x^2\}$. It is clear that $A_0 = A_1 = span\{x^2\}$, $A_2 = \{0\}$ and $\mathcal{B} = \mathcal{A}$. Moreover, $F \in t^2$, $T_1 \notin \mathcal{A}$, and $\mathcal{A}$ does not satisfy the condition (c2) since $T_1(x^2) = 0$. As $\int_{-\varepsilon}^{\varepsilon} (x-0)\, x^i dx = 0$, $i = 0, 2$, then $P_\varepsilon(x) = 0$ is the best approximant to F from $\mathcal{A}$ respect to $\|\cdot\|_\varepsilon^*$. Therefore, the polynomial 0 is the best local approximation to F from $\mathcal{A}$ in 0.*

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