Best local weighted approximation. An approach with abstract seminorms *

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Abstract

In this paper, we extend results given by Zó and Cuenya in 2007 about a general approach to problems of best vector-valued approximation on small regions from a finite dimensional subspace of polynomials of some degree. This approach is called *best local approximation*. We consider a weighted local approximation of a vector valued function on the origin and a weighted best local approximation of a real valued function on several points, similar to classical problems in best local approximation with balanced neighborhood.

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1 Introduction

In 1934, Walsh proved in [13] that the Taylor polynomial of degree l for an analytic function f can be obtained by taking the limit as $\varepsilon \to 0$ of the best Chebyshev approximation to f from π^l on the disk $|z| \leq \varepsilon$, where π^l is the class of polynomials of degree at most l. This paper was the first association between the best local approximation to a function f from π^l in 0 and the Taylor polynomial for f at the origin. However, the concept of best local approximation was introduced by Chui, Shisha, and Smith in [4]. More recently, the best local approximation on several points has been developed in L^p spaces, by [1], [11], [12], [10] and [7], in Orlicz spaces by [9] and with abstract seminorms by [15]. All these studies have considered neighborhoods with the same size on each point. The theory on several points with neighborhoods with different sizes has been introduced by [3] in L^p spaces, and developed by [6], [5] and [8] in Orlicz spaces and by [14] in approaching with abstract seminorms.

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We consider a family of function seminorms $\{\|\cdot\|_\varepsilon\}_{\varepsilon>0}$, acting on Lebesgue measurable functions $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$, where B is the unit ball centered at the origin in $\mathbb{R}^n$. Given a fixed k-tuple of real numbers $\gamma = (\gamma_1, ..., \gamma_k)$, with $\gamma_i > 0$, we denote for each $\varepsilon > 0$, $F^{\varepsilon^\gamma}(x) = (f_1(\varepsilon^{\gamma_1}x), ..., f_k(\varepsilon^{\gamma_k}x))$ and $||F||_\varepsilon^* = ||F^{\varepsilon^\gamma}||_\varepsilon$. For $m \in \mathbb{N} \cup \{0\}$, we denote by π^m the class of algebraic polynomials in n-variables of degree at most m, and $\Pi^{m_1,\cdots,m_k}$ the set $\{P = (p_1, \ldots, p_k) : p_i \in \pi^{m_i}\}$, with $m_i \in \mathbb{N}_0$ for $i = 1, ..., k$. When $m_1 = m_2 = ... = m_k = m$, we write Π_k^m.

Let $\{P_\epsilon\}_{\varepsilon>0}$ be a net of best approximants to F from $\Pi^{m_1,\cdots,m_k}$ with respect to $\|\cdot\|_\varepsilon^*$, i.e., $P_\epsilon \in \Pi^{m_1,\cdots,m_k}$ and

$$\|F - P_\varepsilon\|_\varepsilon^* \le \|F - P\|_\varepsilon^*, \quad \text{for all} \quad P \in \Pi^{m_1,\cdots,m_k}. \tag{1.1}$$

If the net $\{P_\epsilon\}_{\varepsilon>0}$ has a limit as $\epsilon \to 0$, this limit is called the *best local approximation to F from $\Pi^{m_1,\cdots,m_k}$ on the origin.*

In [15], the authors studied the problem (1.1) when $\gamma_1 = \gamma_2 = ... = \gamma_k = 1$ and $m_1 = ... = m_k$. The authors define the class t^m as the functions F which have a Taylor polynomial of degree m at 0 in some sense. They proved the existence of the best local approximation to F in 0, and it was associated with the Taylor polynomial of F at 0.

In this paper, we study the problem (1.1) and we extend the result of [15] to a net of seminorms that depends on the weight parameters $\gamma_1, \gamma_2, ..., \gamma_k$. We consider an approach of a function $F = (f_1, ..., f_k)$ with different weights on each function f_i. For this purpose, we define a new concept of Taylor polynomial of F, whose degree of the i^{th} component is at most $m_i - 1$. We also define a new concept of balanced $k-$tuple $(m_1, ..., m_k)$. This $k-$tuple satisfies a condition relating the order of the Taylor polynomial with the weight $(\epsilon^{\gamma_1}, ..., \epsilon^{\gamma_k})$.

Now, we present an alternative problem to (1.1) that includes classical problems of best multipoint local approximation to a function f when the weight $\epsilon^{\gamma_1} = \epsilon^{\gamma_2} = ... = \epsilon^{\gamma_k}$.

Let $\{x_1, ..., x_k\} \subset [-1, 1]$ and let $\mathcal{M}$ be the set of Lebesgue measurable functions $f : [-2, 2] \subset \mathbb{R} \to \mathbb{R}$. We define

$$Lf(x) := (f(x_1 + x), ..., f(x_k + x)), \quad x \in [-1, 1], \quad f \in \mathcal{M}.$$

Given a positive integer N, we study the best local approximation to Lf from the set $\{Lp : \; p \in \pi^{N-1}\}$ with respect to $\|.\|_\epsilon^*$, i.e., for each $\epsilon > 0$, let p_ϵ be a polynomial in π^{N-1} that minimizes the error

$$||L(f) - L(p)||_\epsilon^*, \tag{1.2}$$

for all $p \in \pi^{N-1}$. If the net $\{p_\epsilon\}_{\epsilon>0}$ converges to a limit in π^{N-1}, this limit is called *best multipoint local approximation of f on $x_1, ..., x_k$.*

This problem, for $\gamma_1 = \gamma_2 = ... = \gamma_k$, has been studied in [12] and [7] for the classical L^p seminorms, in [9] for classical Luxemburg norm and in [15] for abstract seminorms.

In [3], the authors introduced a concept of balanced neighborhoods and balanced integers to solve a problem of best multipoint local approximation in L^p spaces. Later, [6], [5] and [8] extended these studies to Orlicz spaces with Luxemburg seminorms.

In this paper, we study the problem (1.2) which generalizes some results given in [15] to a net of seminorms that depends on the parameters $\gamma_1, \gamma_2, ..., \gamma_k$ to weight the points $x_1, x_2, ..., x_k$ differently. For this purpose, we define a new concept of balanced integers in the context of this work.

This paper is organized as follows. In section 2, we set up conditions over the net of seminorms and define a balanced $k-$tuple, study its properties and give an algorithm which generates them. In section 3, we define a Taylor polynomial of a function on the origin, study its properties and present some preliminary results. In section 4, we show some results on the best local approximation of a function F on the origin, when the approximation class is $\Pi^{m_1,...,m_k}$ and $(m_1, ..., m_k)$ is a balanced k-tuple. Then, we present some preliminary results to extend classical theorems of best multipoint local approximation of a function f. Finally, we solve the problem (1.2).

2 The norm set up and Balanced Integers

Throughout the paper, we assume the following properties for the family of seminorms $\|\cdot\|_\varepsilon$, $0 \le \varepsilon \le 1$. We rewrite these properties from [15, page 195].

(1) For $F = (f_1, \dots, f_k)$ and $G = (g_1, \dots, g_k)$, we have $\|F\|_\varepsilon \le ||G||_\varepsilon$ for every $\varepsilon > 0$, whenever $|f_s| \le |g_s|$, $s = 1, \dots, k$.

(2) If 1 is the function $F(x) = (1, \dots, 1)$, we have $\|1\|_\varepsilon < \infty$ for all $\varepsilon > 0$.

(3) For every $F \in C_k(B)$, we have $\|F\|_\varepsilon \to \|F\|_0$, as $\varepsilon \to 0$, where $C_k(B)$ is the set of continuous functions $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$. Moreover, $\|\cdot\|_0$ is a norm on $C_k(B)$.

A simple example of a net of seminorms fulfilling conditions (1)-(3) is $||F||_\epsilon = \left(\sum\limits_{i=1}^{k} \int_{-1}^{1} |f_i(x)|^{p+\epsilon} \, dx \right)^{1/p+\epsilon}$ for $\epsilon > 0$, with $p > 1$. For other examples of nets of seminorms fulfilling conditions (1)-(3), we refer the reader to [15, section 2].

In particular, when $n = 1$, let us denote the seminorms $||F||_{\epsilon,p} := \left(\sum\limits_{i=1}^{k} \int_{-1}^{1} |f_i(x)|^p \, dx \right)^{1/p}$ for $\epsilon > 0$ and $p > 1$. In this case, the seminorm $\|\cdot\|^*_{\varepsilon,p}$ of a function $F = (f_1, ..., f_k)$ is

$$||F||^*_{\epsilon,p} = \left(\sum_{i=1}^{k} \int\limits_{-\epsilon^{\gamma_i}}^{\epsilon^{\gamma_i}} |f_i(x)|^p \frac{dx}{\epsilon^{\gamma_i}} \right)^{1/p} . \tag{2.1}$$

We will use the net of seminorms $\{\|\cdot\|_{\varepsilon,p}\}_{\varepsilon>0}$ in several examples throughout this paper.

In relation to balanced integers, we can see that in the problem (1.1), $\epsilon^{\gamma_1}, ..., \epsilon^{\gamma_k}$ produce different weights over the functions $f_1, ..., f_k$ in the approximation. On the other hand, in the problem (1.2), $\epsilon^{\gamma_1}, ..., \epsilon^{\gamma_k}$ produce different weights over the points $x_1, ..., x_k$. In this section, we define the balanced integers, which is a concept that depends on weight and approximation class. We give some properties of these integers. Also, we present a form to obtain balanced integers thought an algorithm.

In [3], the authors introduced the concept of balanced integer to solve a problem of best multipoint local approximation in L^p spaces. Now, we define a new concept of balanced integer in the context of this paper.

Given a k-tuple of non negative integers $(m_1, ..., m_k)$ and a k-tuple of real numbers $\gamma = (\gamma_1, ..., \gamma_k)$, with $\gamma_i \in \mathbb{R}^+$, we denote $\gamma_{i_0} m_{i_0} = \min\limits_{1 \le i \le k} \{\gamma_i m_i\}$.

Definition 2.1. *A $k-$tuple of non negative integers $(m_1, ..., m_k)$ is balanced if, for each $m_i > 0$, $\epsilon^{\gamma_{i_0} m_{i_0}} = o(\epsilon^{(m_i - 1)\gamma_i})$, as $\epsilon \to 0$. In this case, we denote $N = \sum\limits_{i=1}^{k} m_i$ as a balanced integer.*

We can see that 0 is a balanced integer. Additionally, under the assumption $\gamma_1 = ... = \gamma_k$, this concept of balanced integer is equivalent to that introduced in [3].

Example 2.2. *Given $\gamma = (3, 2, 1)$, we have that (1,2,3) and (1,3,3) are balanced 3-tuples. So $N = 6$ and $N = 7$ are balanced integers. In fact, $(1, 2, 3)$ is balanced because $\gamma_{i_0} m_{i_0} = 3$, $\epsilon^3 = o(\epsilon^2)$ and $\epsilon^3 = o(1)$ for each $m_i > 0$. Analogously, we can compute that (1,3,3) is balanced.*

Remark 2.3. *It is easy to see that if $\gamma_1 = ... = \gamma_k$, a k-tuple $M = (m_1, ..., m_k)$ is balanced if and only if $M = (m, ..., m)$. In the literature, it is known as the multiple case because $N = \sum_{i=1}^{k} m$ is a multiple of k. In this case, the approximation in the problem (1.1) is from Π_k^m while in the problem (1.2) it is from π^{km-1}, as in [15].*

The balanced integer concept yields a connection between the weight $\epsilon^{\gamma_1}, ..., \epsilon^{\gamma_k}$ and the degree $m_1, ..., m_k$ of the approximation class. This connection allows us to solve the problems (1.1) and (1.2).

Now, we will see that, given $\gamma = (\gamma_1, ..., \gamma_k)$, a balanced integer is uniquely related with a unique balanced k-tuple.

Proposition 2.4. *Given $\gamma = (\gamma_1, ..., \gamma_k)$, to each balanced integer with respect to γ there corresponds exactly one balanced $k-$tuple.*

Proof. Let be $(m_1, ..., m_k)$ a balanced $k-$tuple and $(\overline{m}_1, ..., \overline{m}_k)$ a different k-tuple such that $N = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} \overline{m}_i$.

So, there exist s, j with $m_j \le \overline{m}_j - 1$ and $\overline{m}_s \le m_s - 1$. We denote $\gamma_{i_0} m_{i_0} = \min\limits_{1 \le i \le k} \{\gamma_i m_i\}$ and $\gamma_{i_1} \overline{m}_{i_1} = \min\limits_{1 \le i \le k} \{\gamma_i \overline{m}_i\}$. Then

$$\epsilon^{\gamma_{i_0} m_{i_0}} = o(\epsilon^{\gamma_s (m_s - 1)}) = o(\epsilon^{\gamma_s \overline{m}_s}) = o(\epsilon^{\gamma_{i_1} \overline{m}_{i_1}}).$$

Thus,

$$\frac{\epsilon^{\gamma_{i_1}\overline{m}_{i_1}}}{\epsilon^{\gamma_j(\overline{m}_j-1)}} \geq \frac{\epsilon^{\gamma_{i_1}\overline{m}_{i_1}}}{\epsilon^{\gamma_j m_j}} \geq \frac{\epsilon^{\gamma_{i_1}\overline{m}_{i_1}}}{\epsilon^{\gamma_{i_0} m_{i_0}}} \to \infty,$$

as $\epsilon \to 0$. Therefore, $(\overline{m}_1, ..., \overline{m}_k)$ cannot be a balanced $k-$tuple. $\square$

Proposition 2.5. *Given a $k-$tuple $M = (m_1, ..., m_k)$ with integers $m_i > 0$, there exists a k-tuple $\gamma = (\gamma_1, ..., \gamma_k)$ such that M is a balanced $k-$tuple with respect to γ.*

Proof. In fact, we can consider $\gamma_i = \frac{1}{m_i}$, $1 \leq i \leq k$. $\square$

Throughout the paper, $\gamma = (\gamma_1, \gamma_2, ..., \gamma_k)$ will denote a fixed k-tuple of real numbers with $\gamma_i > 0$. All the balanced k-tuples and balanced integers are balanced with respect to γ. The net of seminorms $\{\|.\|_\epsilon^*\}_{\epsilon>0}$ is also with respect to γ.

The following two results allow us to state an algorithm to compute exactly all the balanced k-tuples. For this purpose, we define the following sets. Given a balanced $k-$tuple $(m_1, ..., m_k)$, set

$$A := \{j : \ \gamma_j m_j = \min_{1\leq i\leq k}\{\gamma_i m_i\}\} \text{ and } B := \{j : \ j \notin A\}. \tag{2.2}$$

Lemma 2.6. *Let $(m_1, ..., m_k)$ and $(q_1, ..., q_k)$ be two balanced k-tuples with $\sum_{i=1}^n m_i < \sum_{k=1}^n q_i$. Consider the sets A and B defined in (2.2) for the $k-$tuple $(m_1, ..., m_k)$. Then*

a) if $j \in A$, then $q_j \geq m_j + 1$;

b) if $j \in B$, then $q_j \geq m_j$.

Proof. Throughout the proof we denote $\gamma_s q_s = \min_{1\leq i\leq k}\{\gamma_i q_i\}$.

a) Suppose to the contrary that $q_j \leq m_j$ for some $j \in A$. By hypothesis, there is an index l such that $q_l \geq m_l + 1$. Then,

$$\frac{\epsilon^{\gamma_s q_s}}{\epsilon^{\gamma_l(q_l-1)}} \geq \frac{\epsilon^{\gamma_j q_j}}{\epsilon^{\gamma_l(q_l-1)}} \geq \frac{\epsilon^{\gamma_j q_j}}{\epsilon^{\gamma_l m_l}} \geq 1 \quad \text{as} \quad \epsilon \to 0,$$

and $(q_1, ..., q_k)$ cannot be balanced. So, we obtain that $q_j \geq m_j + 1$ for all $j \in A$.

b) It is obvious when $m_j = 0$ for all $j \in B$. Now, suppose that $q_j' < m_j$ for some $j \in B$. Then, for $l \in A$, from a) we known that $q_l > 0$ and $q_l - 1 \geq m_l$, so

$$\frac{\epsilon^{\gamma_s q_s}}{\epsilon^{\gamma_l(q_l-1)}} \geq \frac{\epsilon^{\gamma_j q_j}}{\epsilon^{\gamma_l m_l}} \geq \frac{\epsilon^{\gamma_j(m_j-1)}}{\epsilon^{\gamma_l m_l}} \to \infty \quad \text{as} \quad \delta \to 0,$$

where the last condition holds because $(m_1, ..., m_k)$ is balanced. Therefore, $(q_1, ..., q_k)$ cannot be balanced. So, we obtain that $q_j \geq m_j$ for all $j \in B$. $\square$

Given a balanced integer, the above lemma gives us a necessary condition which must satisfy the next balanced integer. The following proposition shows that the equality in this condition generates exactly the following balanced integer.

Proposition 2.7. *Given a balanced* $k-$*tuple* $(m_1, ..., m_k)$*, consider the sets* A *and* B *defined in (2.2). Then, the k-tuple* $(q_1, ..., q_k)$ *defined by* $q_j = m_j + 1$*,* $j \in A$*, and* $q_j = m_j$*,* $j \in B$*, is balanced.*

Proof. Let $\gamma_s q_s = \min_{1\leq i\leq k}\{\gamma_i q_i\}$ and $q_l > 0$. Then

$$\frac{\epsilon^{\gamma_s q_s}}{\epsilon^{\gamma_l (q_l-1)}} = \begin{cases} \frac{\epsilon^{\gamma_s (m_s+1)}}{\epsilon^{\gamma_l m_l}} = \epsilon^{\gamma_s} = o(1), & \text{if } s \in A \text{ and } l \in A; \\ \frac{\epsilon^{\gamma_s (m_s+1)}}{\epsilon^{\gamma_l (m_l-1)}} = \frac{\epsilon^{\gamma_s m_s}}{\epsilon^{\gamma_l (m_l-1)}}\, \epsilon^{\gamma_s} = o(1), & \text{if } s \in A \text{ and } l \in B, \text{ by balanced definition}; \\ \frac{\epsilon^{\gamma_s m_s}}{\epsilon^{\gamma_l m_l}} = o(1), & \text{if } s \in B \text{ and } l \in A; \\ \frac{\epsilon^{\gamma_s m_s}}{\epsilon^{\gamma_l (m_l-1)}} \leq \frac{\epsilon^{\gamma_{i_0} m_{i_0}}}{\epsilon^{\gamma_l (m_l-1)}} = o(1), & \text{if } s \in B \text{ and } l \in b, \text{ where } i_0 \text{ is an index in } A. \end{cases}$$

So, $(q_1, ...q_m)$ is balanced. $\square$

An algorithm was given in [3] which generates all balanced integers in the sense of Chui et. al. in L^p spaces. It was restated and generalized to Luxemburg seminorms in [6] and [5].

Next, we present an algorithm analogous to that in [3], which inductively generates exactly all the balanced k-tuples, i.e., it generates all balanced $k-$tuples $(m_1^{(m)}, ..., m_k^{(m)})$, such that $\sum_{i=1}^k m_i^{(m)}$ is a balanced integer.

Algorithm. Begin with the balanced $k-$tuple $(m_1^{(0)}, ..., m_k^{(0)}) = (0, ..., 0)$ corresponding to the balanced integer 0. Then, in each step, given a k-tuple $(m_1^{(m)}, ..., m_k^{(m)})$ for $m \geq 0$, determine the index set $A_{(m)} = \{j : \ \gamma_j m_j^{(m)} = \min_{1\leq i\leq k}\{\gamma_i m_i^{(m)}\}\}$.
To build the next $k-$tuple $(m_1^{(m+1)}, ..., m_k^{(m+1)})$, put $m_i^{(m+1)} = m_i^{(m)} + 1$, for $i \in A_{(m)}$, and $m_i^{(m+1)} = m_i^{(m)}$, for $i \notin A_{(m)}$.

The super index (m) denote the number of steps in the algorithm minus one.

Lemma 2.8. *The above algorithm generates exactly all balanced k-tuples.*

Proof. From Proposition 2.4, to each balanced integer there corresponds exactly one balanced $k-$tuple. Thus, as a consequence of Lemma 2.6 and Proposition 2.7, an integer N is balanced if only if $N = \sum_{i=1}^n m_i$ for some balanced k-tuple $(m_1, ..., m_k)$ generated by this algorithm. $\square$

Now, we present an example with simple computation to understand the algorithm.

Example 2.9. *Given* $\gamma = (3, 4, 2)$*.*
The algorithm generates the first balanced 3-tuple $(m_1^{(0)}, m_2^{(0)}, m_3^{(0)}) = (0, 0, 0)$*, which corresponds to the balanced integer* $N = 0$*. Then*

$$A_{(0)} = \{j : \ \gamma_j m_j^{(0)} = \min\{\gamma_1 m_1^{(0)}, \gamma_2 m_2^{(0)}, \gamma_3 m_3^{(0)}\}\} = \{j : \ \gamma_j m_j^{(0)} = \min\{0, 0, 0\}\} = \{1, 2, 3\}.$$

The second balanced 3-tuple is $(m_1^{(1)}, m_2^{(1)}, m_3^{(1)}) = (1, 1, 1)$*, which corresponds to the balanced integer* $N = 3$*. Then*

$$A_{(1)} = \{j : \ \gamma_j m_j^{(1)} = \min\{\gamma_1 m_1^{(1)}, \gamma_2 m_2^{(1)}, \gamma_3 m_3^{(1)}\}\} = \{j : \ \gamma_j m_j^{(1)} = \min\{3, 4, 2\}\} = \{3\}.$$

The following balanced 3-tuple is $(m_1^{(2)}, m_2^{(2)}, m_3^{(2)}) = (1, 1, 2)$*, which corresponds to the balanced integer* $N = 4$*. Then*

$$A_{(2)} = \{j : \gamma_j m_j^{(2)} = \min\{\gamma_1 m_1^{(2)}, \gamma_2 m_2^{(2)}, \gamma_3 m_3^{(2)}\}\} = \{j : \gamma_j m_j^{(2)} = \min\{3, 4, 4\}\} = \{1\}.$$

The following balanced 3-tuple is $(m_1^{(3)}, m_2^{(3)}, m_3^{(3)}) = (2, 1, 2)$*, which corresponds to the balanced integer* $N = 5$*. Then*

$$A_{(3)} = \{j : \gamma_j m_j^{(3)} = \min\{\gamma_1 m_1^{(3)}, \gamma_2 m_2^{(3)}, \gamma_3 m_3^{(3)}\}\} = \{j : \gamma_j m_j^{(3)} = \min\{6, 4, 4\}\} = \{2, 3\}.$$

The following balanced 3-tuple is $(m_1^{(4)}, m_2^{(4)}, m_3^{(4)}) = (2, 2, 3)$*, which corresponds to the balanced integer* $N = 7$*. Then*

$$A_{(4)} = \{j : \gamma_j m_j^{(4)} = \min\{\gamma_1 m_1^{(4)}, \gamma_2 m_2^{(4)}, \gamma_3 m_3^{(4)}\}\} = \{j : \gamma_j m_j^{(4)} = \min\{6, 8, 6\}\} = \{1, 3\}.$$

The following balanced 3-tuple is $(m_1^{(5)}, m_2^{(5)}, m_3^{(5)}) = (3, 2, 4)$*, which corresponds to the balanced integer* $N = 9$*. And so on.*

3 The Taylor polynomial

Now, we introduce a generalized version of Taylor polynomial. When $\gamma_1 = ... = \gamma_k$, it generalizes the definition of Taylor polynomial given by A. P. Calderón and A. Zygmund in [2].

Definition 3.1. *Given a k-tuple* $M = (m_1, ..., m_k)$ *with positive integers* m_i*, a function* $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$ *has a Taylor polynomial of degrees* $m_1 - 1, ..., m_k - 1$ *if there exists* $T_{M-1} = T_{M-1}(F) \in \Pi^{m_1-1,...,m_k-1}$ *such that*

$$||F - T_{M-1}||_\varepsilon^* = O(\varepsilon^{\gamma_{i_0} m_{i_0}}),$$

where $\gamma_{i_0} m_{i_0} = \min\limits_{1 \le i \le k} \{\gamma_i m_i\}$*. In this case, we write* $F \in t^{M-1}$*.*

Note that if $\gamma_1 = ... = \gamma_k$ and $m_1 = ... = m_k = m$, the class of function which have a Taylor polynomial in $\Pi^{m_1-1,...,m_k-1}$ is more restrictive than the class given in [15, Definition 3.1]. Therefore, the results we generalize from [15] are generalized considering the class t^{M-1} in the sense of the present work.

In the following example, we present a function F that does not have Taylor polynomial T_{M-1} for some $M > 0$.

Example 3.2. *Let be* $k = 1$ $\gamma = (\gamma_1)$ *and* $M = (1)$*. Then, the function defined by* $F(x) = 1$*, for* $x \ge 0$*, and* $F(x) = -1$*, for* $x < 0$*, does not have any Taylor polynomial in* π^0 *with respect to the seminorms in* L^1 *defined in (2.1). In fact, any polynomial* $T_0(x) = c \in \pi^0$*, with* $c \in \mathbb{R}$*, satisfies that* $\|F - T_0\|_{\epsilon,1}^* = \int_{-\epsilon^{\gamma_1}}^{\epsilon^{\gamma_1}} |F - T_0| \frac{dx}{\epsilon^{\gamma_1}} = C \ne o(\epsilon) = O(\epsilon^{\gamma_{i_0} m_{i_o}})$*, where the constant* C *only depends on* T_0*. So* $F \notin t^{M-1}$*.*

However, under some condition about the function, we can prove the existence of the Taylor polynomial as we present in the following proposition.

Given $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^k$ with $\alpha_i \geq 0$, we denote

$$\partial^\alpha f := \frac{\partial^\alpha f}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_k} x_k}, \quad |\alpha| := \alpha_1 + ... + \alpha_k, \quad x^\alpha := x_1^{\alpha_1} ... x_k^{\alpha_k}.$$

If $F \in t^{M-1}$ and $T = T_{M-1}(F) = \sum_{0 \leq |\alpha| \leq m} C_\alpha x^\alpha$, we set $\partial^\alpha F(0)$ for the vector $\alpha! C_\alpha$ with $\alpha! = \alpha_1! \alpha_2! ..... \alpha_k!$. In particular, if $Lf \in t^{M-1}$ we set $(\partial^\alpha f(x_1), ..., \partial^\alpha f(x_k))$ for tha vector $\alpha! C_\alpha$.

Proposition 3.3. *Given a k-tuple $M = (m_1, ..., m_k)$, if $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$ with $F = (f_1, ..., f_k)$ is a function such that $f_i \in C^{m_i}(B)$, for $i = 1, ..., k$, then $F \in t^{M-1}$.*

Proof. We have that $F^{\varepsilon\gamma}(x) = (f_1(\varepsilon^{\gamma_1} x), ..., f_k(\varepsilon^{\gamma_k} x))$, $x \in B$. Using, for each index i the usual Taylor polynomial of the function f_i of degree $m_i - 1$ around the origin, T_{i,m_i-1}, we obtain that

$$f_i(\varepsilon^{\gamma_i} x) = \sum_{|\alpha|=0}^{m_i-1} \frac{\partial^\alpha f_i(0)}{|\alpha|!} \varepsilon^{\gamma_i |\alpha|} x^\alpha + O(\varepsilon^{\gamma_i m_i}) =: T_{i,m_i-1}(\varepsilon^{\gamma_i} x) + O(\varepsilon^{\gamma_i m_i}), \quad x \in B.$$

Denote $T_{M-1} = (T_{1,m_1-1}, ..., T_{k,m_k-1})$. So, there exists a constant $M > 0$ such that

$$|f_i(\varepsilon^{\gamma_i} x) - T_{i,m_i-1}(\varepsilon^{\gamma_i} x)| \leq M |\varepsilon^{\gamma_i m_i}| \leq M' |\varepsilon^{\gamma_{i_0} m_{i_0}}|, \quad x \in B.$$

Using the monotony of the seminorms $||.||_\varepsilon$,

$$||F^{\varepsilon^\gamma} - T_{M-1}^{\varepsilon^\gamma}||_\varepsilon \leq M' \varepsilon^{\gamma_{i_0} m_{i_0}} ||1||_\varepsilon.$$

So, by the second condition about seminorms,

$$||F - T_{M-1}||_\varepsilon^* = O(\varepsilon^{\gamma_{i_0} m_{i_0}}), \quad \varepsilon \to 0.$$

So $F \in t^{M-1}$ as we required. □

In the following example, we present a function $F \in t^{M-1}$ whose Taylor polynomial is not unique.

Example 3.4. *Given $\gamma = (3, 2)$ and the not balanced $k-$tuple $M = (3, 3)$, let define the function $F = (f_1, f_2)$, with $f_1(x) = x^3$ and $f_2 \in \pi^2$. We will prove that $T = (T_1, f_2)$ and $\overline{T} = (\overline{T}_1, f_2)$ are both Taylor polynomials of F with respect to the seminorms in L^1 defined in (2.1), where $T_1(x) = 0$ and $\overline{T}_1(x) = x^2$. In fact, we must prove that*

$$||F - T_{M-1}||_{\varepsilon,1}^* = \int_{-\epsilon^3}^{\epsilon^3} |f_1 - T_1| \frac{dx}{\epsilon^3} = o(\epsilon^6) \ \ (\ or \int_{-\epsilon^3}^{\epsilon^3} |f_1 - \overline{T}_1| \frac{dx}{\epsilon^3} = o(\epsilon^6)). \tag{3.1}$$

On the other hand, we have that

$$\int_{-\epsilon^3}^{\epsilon^3} |f_1 - T_1| \frac{dx}{\epsilon^3} = \frac{\epsilon^9}{2},$$

and

$$\int_{-\epsilon^3}^{\epsilon^3} |f_1 - \overline{T}_1| \frac{dx}{\epsilon^3} = -\frac{\epsilon^9}{2} + 2\frac{\epsilon^6}{3} \leq 2\frac{\epsilon^6}{3}.$$

So, T and $\overline{T}$ satisfy the condition (3.1).

To prove the uniqueness of the Taylor polynomial we present some auxiliary results. To this purpose we now cite [15, Proposition 3.1].

Proposition 3.5. *There exist $C = C(m, k)$ and $0 < \epsilon(m)$ such that, for all $0 < \epsilon \leq \epsilon(m)$,*

$$C^{-1}||P||_0 \leq ||P||_\epsilon \leq C||P||_0,$$

for all $P \in \Pi_k^m$.

As a consequence we obtain the following Corollary.

Corollary 3.6. *There is a constant $C > 0$ such that*

$$|\partial^\alpha p_i(0)| \leq \frac{C||P^{\epsilon^\gamma}||_\epsilon}{\epsilon^{|\alpha|\gamma_i}},$$

for all $P = (p_1, ..., p_k) \in \Pi^{m_1-1,...,m_k-1}$, $0 \leq |\alpha| \leq m_i - 1$, $1 \leq i \leq k$.

Proof. From Proposition 3.5 we obtain that

$$||P^{\epsilon^\gamma}||_0 \leq C||P^{\epsilon^\gamma}||_\epsilon,$$

for all $P \in \Pi^{m_1-1,...,m_k-1}$ and for all $\epsilon > 0$. Denote $P = (p_1, ..., p_k) \in \Pi^{m_1-1,...,m_k-1}$. So, the function $||P|| := \max_{1\leq i\leq k} \max_{0\leq|\alpha|\leq m_i-1} |\partial^\alpha p_i(0)|$ is a norm on $\Pi^{m_1-1,...,m_k-1}$. Then, from the norm equivalence, using $P^{\epsilon^\gamma}(x) = (p_1(\epsilon^{\gamma_1}x), ..., p_k(\epsilon^{\gamma_k}x))$, we obtain that

$$\max_{1\leq i\leq k} \max_{0\leq|\alpha|\leq m_i-1} |\partial^\alpha p_i(\epsilon^{\gamma_k}x)\ |_{x=0}\ | \leq C||P^{\epsilon^\gamma}||_\epsilon.$$

So

$$\max_{1\leq i\leq k} \max_{0\leq|\alpha|\leq m_i-1} |\epsilon^{\gamma_i|\alpha|}\partial^\alpha p_i(0)| \leq C||P^{\epsilon^\gamma}||_\epsilon,$$

for all $P \in \Pi^{m_1-1,...,m_k-1}$. Then

$$|\epsilon^{\gamma_i|\alpha|}\partial^\alpha p_i(0)| \leq C||P^{\epsilon^\gamma}||_\epsilon,$$

for all $P = (p_1, ..., p_k) \in \Pi^{m_1-1,...,m_k-1}$, $0 \leq |\alpha| \leq m_i - 1$, $1 \leq i \leq k$. □

Now, under some conditions, we prove the uniqueness of the Taylor polynomial of a vector valued function F. Using remark 2.3, the following theorem extends [15, Proposition 3.3].

Theorem 3.7. *Given $M = (m_1, ..., m_k)$ with $m_i > 0$, and $F \in t^{M-1}$, M is a balanced $k-$tuple if and only if the Taylor polynomial of the function F is unique.*

Proof. Let $M = (m_1, ..., m_k)$ be a balanced k-tuple. Suppose that there are two Taylor polynomials of F, T_{M-1} and $\overline{T}_{M-1}$. Then $||T_{M-1} - \overline{T}_{M-1}||_\epsilon^* = O(\epsilon^{\gamma_{i_0} m_{i_0}})$. We denote $T_{M-1} = (p_1, ..., p_k)$ and $\overline{T}_{M-1} = (\overline{p}_1, ..., \overline{p}_k)$. Using Corollary 3.6 for $T_{M-1} - \overline{T}_{M-1}$ and the definition of balanced k-tuple M, we obtain that

$$|\partial^\alpha (p_i - \overline{p}_i)(0)| \leq \frac{C}{\epsilon^{|\alpha|\gamma_i}} ||T_{M-1} - \overline{T}_{M-1}||_\epsilon^* \leq C' \frac{\epsilon^{\gamma_{i_0} m_{i_0}}}{\epsilon^{(m_i-1)\gamma_i}} = o(1),$$

as $\epsilon \to 0$, for $0 \leq |\alpha| \leq m_i - 1$, $1 \leq i \leq k$. Then, the norm

$$||T_{M-1} - \overline{T}_{M-1}|| := \max_{1 \leq i \leq k} \max_{0 \leq |\alpha| \leq m_i - 1} |\partial^\alpha (p_i - \overline{p}_i)(0)| = 0.$$

So, $T_{M-1} = \overline{T}_{M-1}$ as we required. On the other hand, suppose that M is not balanced. Then there exists an index i such that $\frac{\epsilon^{\gamma_{i_0} m_{i_0}}}{\epsilon^{\gamma_i (m_i - 1)}} \neq o(1)$, as $\epsilon \to 0$. Therefore, $\gamma_{i_0 m_{i_0}} \leq \gamma_i (m_i - 1)$. Denote by $T_{M-1} = (p_1, ..., p_k)$ the unique Taylor polynomial of F in $\Pi^{m_1-1,...,m_k-1}$. Define $\overline{T}_{M-1}(x) = (p_1(x), ..., p_i(x) + x^{m_i-1}, ..., p_k(x))$ and $G(x) = (0, ..., x^{m_i-1}, ...0)$, where the monomial x^{m_i-1} is the i^{th} component of G. Then, using properties (1) and (2) of the net $\{\|\|_\epsilon\}_\epsilon$, we obtain $\|G\|_\epsilon^* \leq \|1\|_\epsilon^* \epsilon^{\gamma_i (m_i - 1)}$. Therefore, using $\gamma_{i_0 m_{i_0}} \leq \gamma_i (m_i - 1)$, we obtain

$$\|F - \overline{T}_{M-1}\|_\epsilon^* \leq \|F - T_{M-1}\|_\epsilon^* + \|G\|_\epsilon^* = O(\epsilon^{\gamma_{i_0} m_{i_0}}).$$

So, $\overline{T}_{M-1}$ is another Taylor polynomial of F, which yields a contradiction. □

4 Best Local Approximation

Now, we study the problem of best local approximation (1.1) of a vector-valued function F on the origin. Given a function $F : B \to \mathbb{R}^n$ and a $k-$tuple $M = (m_1, ..., m_k)$, denote by P_ϵ the best approximation to F from $\Pi^{m_1-1,...,m_k-1}$ with respect to the net of seminorms $\|.\|_\epsilon^*$. We can see that P_ϵ exists if $F \in C(B)$, although it can be non unique.

The following theorem proves the existence of the best local approximation of F on the origin and gives a characterization of this approximation. It extends [15, Theorem 3.1] to local approximation with different weights. Denote $M_\epsilon(F)$, for $\epsilon > 0$, as the set of best approximations P_ϵ to F from $\Pi^{m_1-1,...,m_k-1}$.

Theorem 4.1. *Let $M = (m_1, ..., m_k)$ be a balanced $k-$tuple and $F \in t^{M-1}$. Then,*

$$\sup_{P_\epsilon \in M_\epsilon(F)} ||T_{M-1} - P_\epsilon||_{L^\infty(B)} \to 0, \ \text{as } \epsilon \to 0.$$

Proof. Using the triangular inequality, we obtain that $||T_{M-1} - P_\epsilon||_\epsilon^* = O(\epsilon^{\gamma_{i_0} m_{i_0}})$. Denote $P_\epsilon = (p_{\epsilon,1}, ..., p_{\epsilon,k})$ and $T_{M-1} = (p_1, ..., p_k)$ as two $k-$tuples of polynomials in $\Pi^{m_1-1,...,m_k-1}$. From Corollary

3.6, there exist constants $C = C(k, m)$ and C' such that

$$|\partial^\alpha(p_{\epsilon,i} - p_i)| \leq \frac{C\ ||T_{m-1} - P_\epsilon||^*_\epsilon}{\epsilon^{|\alpha|\gamma_i}} \leq C' \frac{\epsilon^{\gamma_{i_0} m_{i_0}}}{\epsilon^{(m_i-1)\gamma_i}} o(1),$$

as $\epsilon \to 0$, for $0 \leq |\alpha| \leq m_i - 1$, $1 \leq i \leq k$. Then

$$||T_{m-1} - P_\epsilon|| := \max_{1\leq i\leq k} \max_{0\leq|\alpha|\leq m_i-1} |\partial^\alpha(p_{\epsilon,i} - p_i)| = o(1),$$

as $\epsilon \to 0$. So, by the norm equivalence on $\Pi^{m_1-1,\ldots,m_k-1}$, we obtain $||T_{M-1} - P_\epsilon||_{L^\infty(B)} \to 0$, as $\epsilon \to 0$. Since the constants C, C' and the equivalence constant only depend on m, k, T_{M-1}, the convergence does not depend on the election of P_ϵ. Therefore,

$$\sup_{P_\epsilon \in M_\epsilon(F)} ||T_{m-1} - P_\epsilon||_{L^\infty(B)} \to 0, \text{ as } \epsilon \to 0.$$

□

Remark 4.2. *Under the hypothesis of Theorem 4.1, T_{M-1} is the best local approximation to F on the origin and the convergence is uniform and independent from the selection of the net $\{P_\epsilon\}_{\epsilon>0}$.*

Example 4.3. *Given $\gamma = (3, 2)$ and the balanced $k-$tuple $M = (1, 1)$, we define the function $F = (f_1, f_2)$, with $f_1(x) = \frac{5}{3}x$ for $x \geq 0$, $f_1(x) = x$ for $x < 0$, and $f_2(x) = \frac{1}{2}$ for $x \in \mathbb{R}$. We will prove that $T = (T_1, T_2)$ with $T_1(x) = 0$ and $T_2(x) = \frac{1}{2}$, for $x \in [-1, 1]$ is the Taylor polynomial of F with respect to the seminorms in L^1 defined in (2.1). In fact, $T \in \Pi_2^0$ and we prove that*

$$||F - T||^*_{\varepsilon,1} = \int_{-\epsilon^3}^{\epsilon^3} |f_1| \frac{dx}{\epsilon^3} = \int_0^{\epsilon^3} \frac{5}{3} x \frac{dx}{\epsilon^3} + \int_{\epsilon^3}^0 |x| \frac{dx}{\epsilon^3} = \frac{5}{4}\epsilon^2 + \frac{1}{2}\epsilon^2 = O(\epsilon^2).$$

Then, from Theorem 4.1, $P_\epsilon \to T$ as $\epsilon \to 0$, i.e., T is the best local approximation of F on the origin with respect to the seminorm $\|\cdot\|^*_{\epsilon,1}$.

Remark 4.4. *We observe that, given a polynomial $P = (p_1, ..., p_k) \in \Pi^{m_1-1,\ldots,m_k-1}$, whose component p_i interpolates the derivatives $f_i^{(j)}(0)$, $0 \leq j \leq m_i - 1$, for $1 \leq i \leq k$, of a certain function $F = (f_1, ..., f_k)$. From Proposition 2.5, there exists a scaling $(\epsilon^{\gamma_1}, ..., \epsilon^{\gamma_k})$ such that $M = (m_1, \ldots, m_k)$ is a balanced k-tuple. Using the proof of Proposition 3.3, $P \in \Pi^{m_1-1,\ldots,m_k-1}$ is the Taylor polynomial of F. Then, using the above theorem P is the best local approximation of F from $\Pi^{m_1-1,\ldots,m_k-1}$ on the origin with respect to $\|\cdot\|^*_\epsilon$.*

From now on, we will study the problem (1.2). The following example shows that the best multi-point local approximation of a function may not exist.

Example 4.5. *Let $x_1 = 0$, $x_2 = 1$, $\gamma = (2, 1)$. We consider the problem (1.2) approaching a function f with respect to the seminorms in L^1 defined in (2.1). The function f is defined as $f(x) = 0$, for*

$x \leq 1/2$, and $f(x) = 1$, for $x > 1/2$. Let $N = 1$ be a not balanced integer, i.e., we must approximate f from constant polynomials. It is easy to see that the best approximation $p_\epsilon \equiv a$ satisfies that $0 \leq a \leq 1$. For each $\epsilon > 0$, we have to minimize

$$\sum_{i=1}^{2} \int_{x_i-\epsilon^{\gamma_i}}^{x_i+\epsilon^{\gamma_i}} |f(x) - a| \frac{dx}{\epsilon^{\gamma_i}} = \int_{-\frac{\epsilon^2}{2}}^{\frac{\epsilon^2}{2}} a \ \frac{dx}{\epsilon^2} + \int_{1-\frac{\epsilon}{2}}^{1+\frac{\epsilon}{2}} (1-a)\frac{dx}{\epsilon} = 1,$$

for all $0 \leq a \leq 1$. Then, we obtain that all the polynomials $p_\epsilon = a$, with $0 \leq a \leq 1$, are the best approximations of the function f for each $\epsilon > 0$. Thus there is no best multipoint local approximation of f.

Now, we state some definitions and preliminary results to solve problem (1.2). The following proposition extends [15, Proposition 3.6].

Proposition 4.6. *Let $M = (m_1, ..., m_k)$ and $N = \sum_{i=1}^{k} m_i$. The function $Lf \in t^{M-1}$ if and only if there exists a polynomial $h \in \pi^{N-1}$ such that $||L(f) - L(h)||_\epsilon^* = O(\epsilon^{\gamma_{i_0} m_{i_0}})$, as $\epsilon \to 0$. Moreover, if there exists h, it is unique and it is called the Hermite interpolation polynomial since it interpolates the data $f^{(j)}(x_i)$ for $0 \leq j \leq m_i - 1$, $1 \leq i \leq k$.*

Proof. If $P \in \Pi^{m_1-1,\dots,m_k-1}$, there exists a unique polynomial $h \in \pi^{N-1}$ which interpolates $h^\alpha(x_i) = p_i^\alpha(0)$, $0 \leq \alpha \leq m_i - 1$, $1 \leq i \leq k$, where $P = (p_1, ..., p_k)$. Then, the function $I : \Pi^{m_1-1,\dots,m_k-1} \to \pi^{N-1}$, with $I(P) = h$, is an isomorphism. Furthermore,

$$||P - L(h)||_\epsilon^* = O(\epsilon^{\gamma_{i_0} m_{i_0}}).$$

In fact, we know that $(P - L(h))^{\epsilon^\gamma}(x) = (p_1(\epsilon^{\gamma_1} x) - h(x_1 + \epsilon^{\gamma_1} x), ..., p_k(\epsilon^{\gamma_k} x) - h(x_k + \epsilon^{\gamma_k} x))$, $x \in [-1, 1]$. Using the usual Taylor polynomial of h on the origin, for each index i, we obtain $p_i(\epsilon^{\gamma_i} x) = \sum_{|\alpha|=0}^{m_i-1} \frac{p_i^\alpha(0)}{|\alpha|!} \epsilon^{\gamma_i |\alpha|} x^\alpha = \sum_{|\alpha|=0}^{m_i-1} \frac{h^\alpha(x_i)}{|\alpha|!} \epsilon^{\gamma_i|\alpha|} x^\alpha = h(x_i + \epsilon^{\gamma_i} x) - O(\epsilon^{\gamma_i m_i})$. Then, $|p_i(\epsilon^{\gamma_i} x) - h(x_i + \epsilon^{\gamma_i} x)| = O(\epsilon^{\gamma_{i_0} m_{i_0}})$, $1 \leq i \leq k$. Therefore, $||P - L(h)||_\epsilon^* \leq \overline{M} \epsilon^{\gamma_{i_0} m_{i_0}} ||1||_\epsilon$ as we required.

Now, if $h \in \pi^{N-1}$ satisfies that $||L(f) - L(h)||_\epsilon^* = O(\epsilon^{\gamma_{i_0} m_{i_0}})$, using triangular inequality, we obtain $||L(f) - P||_\epsilon^* = O(\epsilon^{\gamma_{i_0} m_{i_0}})$, where $P \in \Pi^{m_1-1,\dots,m_k-1}$ such that $I(P) = h$, i.e., $L(f) \in t^{M-1}$. If the last condition holds, using triangular inequality, we obtain $||L(f) - L(h)||_\epsilon^* = O(\epsilon^{\gamma_{i_0} m_{i_0}})$ for $h = I(P)$. $\square$

The following result shows the existence and characterization of the best multipoint local approximation of f from π^{N-1}. It extends the work in [15, page 204] about the problem (1.2) with respect to the seminorms in L^p defined in (2.1) when $\gamma_1 = ... = \gamma_k$. We denote by $m_\epsilon(f)$ the set of best approximation p_ϵ of f from π^{N-1} on $x_1, ..., x_k$ with respect the net $\|.\|_\epsilon^*$.

Theorem 4.7. *If $M = (m_1, ..., m_k)$ is a balanced $k-$tuple, $N = \sum_{i=1}^{k} m_i$ and $Lf \in t^{M-1}$, there exists a polynomial $h \in \pi^{N-1}$ such that*

$$\sup_{p_\epsilon \in m_\epsilon(f)} ||p_\epsilon - h||_{L^\infty([-1,1])} \to 0, \ \textit{as } \epsilon \to 0.$$

Proof. From Proposition 4.6, there exists a polynomial $h \in \pi^{N-1}$ with the condition $||L(f)-L(h)||_\epsilon^* = O(\epsilon^{\gamma_{i_0} m_{i_0}})$. Using triangular inequality and the definition of best approximation, we obtain $||L(p_\epsilon) - L(h)||_\epsilon^* = O(\epsilon^{\gamma_{i_0} m_{i_0}})$. Since $L(h-p_\epsilon) \in \Pi_k^{N-1}$, from Proposition 3.6, we obtain that

$$|\partial^\alpha (h-p_\epsilon)(x_i)| \leq \frac{C(N,k)}{\epsilon^{\alpha\gamma_i}} ||L(p_\epsilon) - L(h)||_\epsilon^*, \text{ for } 0 \leq \alpha \leq N-1,\ 1 \leq i \leq k.$$

In particular, this occurs in $0 \leq \alpha \leq m_i - 1$. Using $\epsilon^{\gamma_i m_i - 1} \leq \epsilon^{\gamma_i \alpha}$ and the definition of balanced $k-$tuple, we obtain that $|\partial(h-p_\epsilon)(x_i)| = o(1)$, as $\epsilon \to 0$, for $0 \leq \alpha \leq m_i - 1,\ 1 \leq i \leq k$. Therefore,

$$||h-p_\epsilon|| := \max_{1\leq i\leq k} \max_{0\leq\alpha\leq m_i-1} |\partial^\alpha (h-p_\epsilon)(x_i)| = o(1),$$

as $\epsilon \to 0$. Now, from the equivalence of norms, since the constants used are independent from the selected net $\{p_\epsilon\}$, we obtain the required result. $\square$

Remark 4.8. *In particular, under the hypothesis of Theorem 4.7, there exists the best multipoint local approximation of f from π^{N-1}. It is the polynomial $h \in \pi^{N-1}$ which interpolates the data $f^\alpha(x_i)$, $0 \leq \alpha \leq m_i - 1$, $1 \leq i \leq k$. Moreover, the convergence is uniform on $[-1,1]$, and independent from the selection of $p_\epsilon \in m_\epsilon(f)$.*

Example 4.9. *Let us consider $x_1 = 0$, $x_2 = 1$, $\gamma = (2,1)$, the balanced 2-tuple $M = (1,1)$ and the balanced integer $N = 2$. The function f is defined as $f(x) = 0$, for $x \leq 1/2$, and $f(x) = \frac{1}{2}$, for $x > 1/2$. Then $Lf(x) = (f(0+x), f(1+x)) = (3, \frac{1}{2})$, for $x \in (-\frac{1}{2}, \frac{1}{2})$. It is easy to see that $Lf \in t^{M-1}$ with Taylor polynomial $T = (T_1, T_2) = (0, \frac{1}{2})$. From the proof of Proposition 4.6 and Theorem 4.7, the polynomial $h \in \pi^1$ and $h(x_1) = T_1(0)$ and $h(x_2) = T_2(0)$. Thus, $h(x) = \frac{1}{2}x$ and $p_\epsilon \to h$, as $\epsilon \to 0$, i. e., h is the best multipoint local approximation of f from π^1 with respect the seminorms $||.||_\epsilon^*$.*

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