EXISTENCE AND CHARACTERIZATION OF BEST $\varphi-$ APPROXIMATIONS BY LINEAR SUBESPACES

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Abstract. Let L^φ be an Orlicz space. We give sufficient conditions on φ to ensure that there exists a best φ-approximation from a linear subspace $S \subset L^\varphi$. In addition, given an non necessarily linear operator T defined from L^φ into itself, we give necessary and sufficient conditions on T to ensure that this is a best φ-approximation operator from a linear subspace S.

1. Introduction and notations

Let $\mathcal{F}$ be the class of all non decreasing functions φ defined for all real numbers $t \geq 0$, with $\varphi(0^+) = 0$ and $\varphi(\infty) = \infty$. We also assume a Δ_2 condition for the functions φ, which means that there exists a constant $\Lambda = \Lambda_\varphi > 0$ such that $\varphi(2a) \leq \Lambda\varphi(a)$, for $a \geq 0$.

Let $\varphi \in \mathcal{F}$ and let $(B, \mathcal{A}, m)$ be the Lebesgue measure space where $B \subset \mathbb{R}$ is a bounded set. We denote $L^\varphi = L^\varphi(\mathcal{A}) = L^\varphi(B, \mathcal{A}, m)$ by the class of all $\mathcal{A}$-measurable functions f defined on B such that $\int_B \varphi(|f|)dx < \infty$.

Given a set $S \subset L^\varphi$, an element $s^* \in S$ is called *a best φ-approximation of $f \in L^\varphi$ from the approximation class S* if and only if

$$\int_B \varphi(|f - s^*|)dm = \inf_{s\in S} \int_B \varphi(|f - s|)dm =: E_S(f),$$

and, in this case, we write $s^* \in \mu_\varphi(f/S)$. The mapping $\mu_\varphi : L^\varphi \to 2^S$ is called the *best φ-approximation operator* from S.

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In the present paper we consider two problems. The study of existence of the best φ-approximation (i.e. $\mu_\varphi(f/S) \neq \emptyset$) and the characterization of all operators that behaves as the best φ-approximation operator μ_φ defined before. Both problems have been studied in different contexts depending on the function φ and the structure of the approximation class S.

The problem of existence was extensively treated in [13] for the case where S is a lattice, (i.e. if $f, g \in S$, then $min(f,g) \in S$ and $max(f,g) \in S$), obtained by a sub $\sigma-$ algebra $\mathcal{L}$ of measurable functions in $L^\varphi(\mathcal{A})$ denoted by $L^\varphi(\mathcal{L})$. Many cases of best approximation by linear subspaces are embrace with this setup, for instance, the classical conditional expectation with a sub $\sigma-$algebra or the case $\varphi(t) = t^2$ with a sub $\sigma-$lattice which was primary considered in [3]. Also [13] covers the case for $\varphi(x) = x^p,\ \ p > 1$ and a sub $\sigma-$ algebra $\mathcal{L}$. The concept of $p-$ predictors that was treated by Ando and Amemiya in [1], and with sub $\sigma-$ lattices treated in the seventies in [2]. Another case covered by [13] is when $\varphi(x) = x$ and considering sub $\sigma-$ algebras which is the concept of conditional medians presented in [19]. We can mention the study of the extended best approximation operator when the class S comes from a $\sigma-$ lattice, see [10].

In all these cases the approximation classes were suitable lattices, and most of these cases were treated for $\varphi(x) = x^p$.

In this paper, we obtain that $\mu_\varphi(f/S) \neq \emptyset$ for a rather general non decreasing function φ as the ones considered in [13] and with a finite- dimensional linear subspace of L^φ as the approximation class. For example the real polynomials defined on B with degree at most n, which is not necessarily a lattice. This result is presented in Section 2.

Concerning the characterization problem, for the best linear approximant operator , it has been investigated by many authors. For the case of the classical conditional expectation, i.e, $\phi(t) = t^2$ and where S is a set of measurable functions respect to a sub σ-algebra, the first general result appears in [15]. A similar characterization result was given in [5]. These results were also treated by other authors in [8, 16, 17, 18] around the sixties.

A characterization for a non linear operator as a conditional expectation with respect to a sub σ-lattice appears in [9]. In [12] it was given a characterization of the best approximation operator from a sub σ-lattice in L^p space, $1 < p < \infty$. The same authors in [14] extended these results considering Orlicz spaces L^φ and also they got a characterization of best approximation operator for $\sigma-$ algebras as best approximation classes. In [6] the authors study the characterization of the extended best φ-approximation operator.

In this paper we obtain, in Theorem 3.4 and Theorem 3.6, a characterization of the best approximation operators considering linear subspaces of L^φ. Also in Theorem 3.12 we get a characterization of best approximation operators considering fewer requirements on φ, to those in [14], and different hypothesis on the operator T to ensure that it is a best approximation operator from a sub $\sigma-$ algebra.

2. Existence of best approximations

In this section we prove the existence of the best approximation when the class S is a finite-dimensional subspace from L^φ. For this purpose, we first prove the following lemma.

Lemma 2.1. *Let $\varphi \in \mathcal{F}$ be a left continuous function and $f \in L^\varphi$. Let $S \subset L^\varphi$ be a finite-dimensional subspace. Suppose there exist $K > 0$ and a sequence $\{s_k\}_{k\in\mathbb{N}} \subset S$ such that*

$$\int_B \varphi(|f - s_k|)dm \leq K, \quad \text{for all } k \in \mathbb{N}.$$

If $\{s_k\}_{k\in\mathbb{N}} \subset L^\infty(B)$, then there exists $M > 0$ such that $\|s_k\|_{L^\infty(B)} \leq M$, for all $k \in \mathbb{N}$.

Proof. Suppose that there exists a subsequence $\left\{\|s_{k_j}\|_\infty\right\}_{j\in\mathbb{N}}$ such that $\|s_{k_j}\|_\infty \neq 0$ and $\lim_{j\to\infty} \|s_{k_j}\|_{L^\infty(B)} = \infty$. Since $\{g \in L^\infty(B) : \|g\|_{L^\infty(B)} \leq 1\} \cap S$ is compact, there exists $g \in S$ such that $\lim_{j\to\infty} \left\| \frac{s_{k_j}}{\|s_{k_j}\|_{L^\infty(B)}} - g \right\|_{L^\infty(B)} = 0$ and $\|g\|_{L^\infty(B)} = 1$. Let us define

$$A_j(\lambda) := \left\{ \left|f - s_{k_j}\right| > \lambda \, \|s_{k_j}\|_{L^\infty(B)} \right\}.$$

We claim that there exist λ, $\alpha > 0$ and a subsequence, which we will also denote by $\{s_{k_j}\}$, such that $m(A_j(\lambda)) > \alpha$ for all $j \in \mathbb{N}$.

Indeed, if $\lim_{j\to\infty} m(A_j(\lambda)) = 0$ for all $\lambda > 0$, then $\frac{f}{\|s_{k_j}\|_{L^\infty(B)}} - \frac{s_{k_j}}{\|s_{k_j}\|_{L^\infty(B)}}$ converges to 0 in measure. Then, there exists another subsequence, denoted in the same way, such that $\frac{f}{\|s_{k_j}\|_{L^\infty(B)}} - \frac{s_{k_j}}{\|s_{k_j}\|_{L^\infty(B)}}$ converges to 0 a.e. on B. This implies that $\frac{f}{\|s_{k_j}\|_{L^\infty(B)}}$ converges to g a.e. on B. Thus $g = 0$ a.e. on B and this contradicts that $\|g\|_{L^\infty(B)} = 1$.

Now, we have

$$\alpha\varphi(\lambda \|s_{k_j}\|_\infty) \le m(A_j(\lambda))\varphi(\lambda \|s_{k_j}\|_\infty) \le \int_{A_j(\lambda)} \varphi(\lambda \|s_{k_j}\|_\infty) dm \le \int_B \varphi(|f - s_{k_j}|) dm \le K.$$

Finally, letting $j \to \infty$, we obtain a contradiction. □

The following example shows that the above lemma does not remain valid if the assumption $\{s_k\}_{k\in\mathbb{N}} \subset L^\infty(B)$ is not required.

Example 2.2. *Let $B = [0,1]$, $\varphi(x) = x^{\frac{1}{2}}$, $f(x) = 0$, and let S be the subspace spanned by the function $g(x) = \frac{1}{x}$. Clearly the sequence $\{s_k\}_{k\in\mathbb{N}} \subset L^\varphi$ defined by $s_k = \frac{1}{k}g$ satisfies*

$$\int_B \varphi(|s_k|) dm \le \int_B \varphi(|g|) dm.$$

However, $\|s_k\|_{L^\infty(B)} = \infty$ for all $k \in \mathbb{N}$.

Theorem 2.3. *Let $\varphi \in \mathcal{F}$ be a left continuous function and let $S \subset L^\varphi \cap L^\infty(B)$ be a finite-dimensional subspace. Then for $f \in L^\varphi$ there exists $s^* \in S$ such that*

$$\int_B \varphi(|f - s^*|) dm = \inf_{s\in S} \int_B \varphi(|f - s|) dm = E_S(f).$$

Proof. Let $\{s_k\}_{k\in\mathbb{N}} \subset S$ be such that $\int_B \varphi(|f - s_k|) dm \le E_S(f) + \frac{1}{k}$. From Lemma 2.1 $\left\{\|s_k\|_{L^\infty(B)}\right\}_{k\in\mathbb{N}}$ is bounded. So, there exists $s^* \in S$ such that $\lim_{j\to\infty} \|s_{k_j} - s^*\|_{L^\infty(B)} = 0$ for some subsequence $\{s_{k_j}\}_{j\in\mathbb{N}}$. Then $s_{k_j} - s^*$ converges to 0 a.e. on B. Finally, by Fatou's Lemma and the left continuity for φ, we have

$$\int_B \varphi(|f - s^*|) dm \le \int_B \liminf_{j\to\infty} \varphi(|f - s_{k_j}|) dm \le \liminf_{j\to\infty} \int_B \varphi(|f - s_{k_j}|) dm \le E_S(f).$$

□

Remark 2.4. *Note that in the proof of Theorem 2.3, to get the sequence $\left\{\|s_k\|_{L^\infty(B)}\right\}_{k\in\mathbb{N}}$ bounded, we can use that $\int_B \varphi(|f - s_k|) dm \le K$ implies that $\int_B \varphi(|s_k|) dm \le M$. Now, if we* *consider as the approximation class the real polynomials P of degree at most n, and if φ is also a continuous function, we can have an easier proof to get a uniform the bound of $\|s_k\|_\infty$. To do this, we define $F : (\Pi^n, \|\cdot\|_{L^\infty(B)}) \to \mathbb{R}$ by $F(P) = \int_B \varphi(|P|)dm$, then F is continuous and reaches the maximum and minimum in the compact set $\left\{\frac{P}{\|P\|_{L^\infty(B)}} : P \in \Pi^n, P \neq 0\right\}$. So, there exist $C, D > 0$ such that*

$$C \leq F\left(\frac{P}{\|P\|_{L^\infty(B)}}\right) \leq D, \quad \text{for all} \quad P \in \Pi^n.$$

In particular, if $\varphi = x^q$, $0 < q < \infty$, then $C\|P\|^q_{L^\infty(B)} \leq \int_B |P|^q dm = F(P)$, thus the upper bound of $F(P)$ implies an upper bound of $\|P\|_{L^\infty(B)}$.

In the case that φ is a convex function, by the Δ_2 condition on φ and Jensen's inequality we have

$$\varphi(\|P\|_{L^1(B)}) = \varphi\left(\int_B |P|dm\right) \leq K\int_B \varphi(|P|)dm.$$

Therefore, if $\int_B \varphi(|P|)dm \leq M$, then $\varphi(\|P\|_{L^1(B)}) \leq M'$ for some positive M'. Using that $\varphi(\infty) = \infty$ we have that there exists M such that $\|P\|_{L^1(B)} \leq M$, for all $P \in \Pi^n$.

In the next example, we show that the best polynomial approximation may not be unique.

Example 2.5. *Let $B = [0, 1]$ and $S = \Pi^0$ be the set of all constant functions and let $f(x) = \chi_{[0,1/2]}(x)$ and $\varphi(x) = \sqrt{x}$. Then a direct calculation gives $\mu_\varphi(f/S) = \{0, 1\}$.*

The next example shows that the left continuity condition on φ is not superfluous.

Example 2.6. *Let φ be the following non decreasing and non left continuous function*

$$\varphi(x) = \begin{cases} x, & \text{if} \quad 0 \leq x < \frac{1}{2} \\ 2x + 1 & \text{if} \quad \frac{1}{2} \leq x \leq 1. \end{cases}$$

Set $B = [0, 1]$, $f(x) = \chi_{[0,\frac{1}{2}]}(x)$ and, as the approximation class $S = \Pi^0$. Then the approximation problem become to minimize the function

$$F(x) = \varphi(x) + \varphi(1 - x),$$

for $0 \leq x \leq 1$. Then $F(x) = -x + 3$, for $x \neq \frac{1}{2}$, and $F(\frac{1}{2}) = 4$, and the minimum is not reached.

Proposition 2.7. *Let $\varphi \in \mathcal{F}$ be a left continuous function and let $S \subset L^\varphi \cap L^\infty(B)$ be a finite-dimensional subspace. Then $\mu_\varphi(f/S)$ is a compact set in $(S, \|\cdot\|_{L^\infty(B)})$.*

Proof. Let $\{s_k\}_{k\in\mathbb{N}} \subset \mu_\varphi(f/S)$ be such that $\int_B \varphi(|f-s_k|)dm = E_S(f)$. According to Lemma 2.1 we obtain that $\left\{\|s_k\|_{L^\infty(B)}\right\}_{k\in\mathbb{N}}$ is bounded. So, there exist $s^* \in S$ and a subsequence of $\{s_k\}_{k\in\mathbb{N}}$, denoted again by $\{s_k\}$, such that $\lim_{k\to\infty} \|s^* - s_k\|_{L^\infty(B)} = 0$. Using the same argument as in the end of the proof of Theorem 2.3, we have that $s^* \in \mu_\varphi(f/S)$, which means that $\mu_\varphi(f/S)$ is sequentially compact in $(S, \|\cdot\|_{L_\infty(B)})$ and this finishes the proof. □

3. Characterization of Best φ-approximation operators

For $C \subset L^\varphi$, we write by $\overline{C}^\varphi$ by the set of all limits of sequences on C where the convergence means to consider

$$d_\varphi(f,g) = \int_B \varphi(|f-g|)dm.$$

Let $T : L^\varphi \to L^\varphi$ be a single-valued operator. We denote the range of T by

$$R_T = \{T(f) : f \in L^\varphi\}.$$

In this section we give necessary and sufficient conditions on an operator T to assure that it is a best approximation operator. With this aim we introduce the following definition.

Definition 3.1. *A single-valued operator $T : L^\varphi \to L^\varphi$ is called*

(a) quasiadditive if $T(f+Tg) = Tf + Tg$, for all $f, g \in L^\varphi$;

(b) quasihomogeneous if $T(\alpha Tf) = \alpha Tf$, for all $f \in L^\varphi$ and $\alpha \in \mathbb{R}$;

(c) quasialgebraic if $T(1) = 1$ and $T(Tg\,Tf) = Tg\,Tf$, for all $f, g \in L^\infty(B)$;

(d) φ-closed if $\lim_{n\to\infty} d_\varphi(f_n, f) = 0$ and $\lim_{n\to\infty} d_\varphi(Tf_n, g) = 0$, then $Tf = g$;

(e) φ-expectation invariant if $\int_B \varphi(|f-Tf|)dm \le \int_B \varphi(|f|)dm$ for $Tf \neq 0$;

(f) φ-bounded if $R_T \subset \overline{R_T \cap L^\infty(B)}^\varphi$.

The following examples show that the best approximation operators satisfy most of the conditions given in the last definition.

Example 3.2. *It is easy to see that*

(i) If $\varphi(t) = t^p$, $1 < p < \infty$, and S is the class of algebraic polynomials with real coefficients of degree at most n, then the operator T defined by $T(f) = \mu_\varphi(f/S)$, $f \in L^p$, satisfies (a),(b) and (e) of Definition 3.1.

(ii) If $\varphi(t) = t^p$, $1 < p < \infty$, $B = [0, 1]$, and $S = span\left\{\chi_{[0,\frac{1}{2})}, \chi_{[\frac{1}{2},1]}\right\}$, then the operator T defined by $T(f) = \mu_\varphi(f/S)$, satisfies (a)-(f) of Definition 3.1.

Next we need some auxiliary results.

Lemma 3.3. *Let $T : L^\varphi \to L^\varphi$ be a single-valued operator. If T is quasiadditive and $T0 = 0$, then $R_T = \{f \in L^\varphi \ : \ Tf = f\}$. In addition, if T is φ-closed then $\overline{R_T}^\varphi = R_T$.*

Proof. Set $S = \{f \in L^\varphi \ : \ Tf = f\}$. Clearly $S \subset R_T$. On the other hand, since T is quasiadditive and $T(0) = 0$, then

$$T(Tf) = Tf, \quad \text{for all} \quad f \in L^\varphi.$$

Therefore $R_T \subset S$.

Now, let $g \in \overline{R_T}^\varphi$. Then there exists a sequence $\{g_k\}_{k\in\mathbb{N}} \subset R_T$ such that $\lim_{k\to\infty} d_\varphi(Tg_k, g) = 0$. Since $T(Tg_k) = Tg_k$, $\lim_{k\to\infty} d_\varphi(T(Tg_k), g) = 0$. As T is φ-closed, we have $Tg = g$, and so $g \in R_T$. □

The following theorem gives sufficient conditions to assure an operator T is a best approximation operator.

Theorem 3.4. *Let $\varphi \in \mathcal{F}$ and let $T : L^\varphi \to L^\varphi$ be a single-valued operator. If T is quasiadditive, quasihomogeneous and φ-expectation invariant, then $Tf \in \mu_\varphi(f/R_T)$ for all $f \in L^\varphi$.*

Proof. We have $T0 = 0$, since T is quasihomogeneous. So, Lemma 3.3 shows that $R_T = \{f \in L^\varphi \ : \ Tf = f\}$. As T is quasiadditive, we obtain

$$Tf - Tg = Tf + T(-Tg) = T(f + T(-Tg)) = T(f - Tg), \quad f, g \in L^\varphi. \tag{1}$$

Let $f \in L^\varphi$ and $P \in R_T \setminus \{Tf\}$. Since $TP = P$, according to (1), we have $T(f - TP) = Tf - TP = Tf - P \neq 0$. As T is φ-expectation invariant, we get

$$\begin{aligned}\int_B \varphi(|f - Tf|)dm &= \int_B \varphi(|f - TP - (Tf - TP)|)dm \\ &= \int_B \varphi(|f - TP - T(f - TP)|)dm \\ &\leq \int_B \varphi(|f - TP|)dm = \int_B \varphi(|f - P|)dm.\end{aligned}$$

Therefore $Tf \in \mu_\varphi(f/R_T)$. □

Remark 3.5. *Under the same hypotheses of Theorem 3.4, if $\mu_\varphi(f/R_T)$ is a singleton for all $f \in L^\varphi$, then $Tf = \mu_\varphi(f/R_T)$ for all $f \in L^\varphi$. This is the case, for example if the definition 3.1 (e) has a strict inequality.*

If we also assume the uniqueness of the best approximation for all $f \in L^\varphi$, we get the following characterization result.

Theorem 3.6. *Let $\varphi \in \mathcal{F}$ and let $T : L^\varphi \to L^\varphi$ be a single-valued operator. Assume that $\mu_\varphi(f/R_T)$ is a singleton for all $f \in L^\varphi$. Then the following statements are equivalents:*

(a) R_T is a subspace of L^φ, and $Tf = \mu_\varphi(f/R_T)$ for all $f \in L^\varphi$.

(b) T is quasiadditive, quasihomogeneous, and φ-expectation invariant.

Proof. $(a) \Rightarrow (b)$ Let $f, g \in L^\varphi$ and $\alpha \in \mathbb{R}$. Clearly T is φ-expectation invariant. Since $\int_B \varphi(|\alpha f - \alpha T(f)|)dm = 0$ for $f \in R_T$, then T is quasihomogeneous. Finally, an easy computation shows that $Tf + Tg = \mu_\varphi((f + Tg)/R_T)$ and consequently $T(f + Tg) = Tf + Tg$, i.e., T is quasiadditive.

$(b) \Rightarrow (a)$ Let $f \in L^\varphi$. By hypothesis and Theorem 3.4 we have

$$Tf = \mu_\varphi(f/R_T).$$

Now, we claim that R_T is a subspace of L^φ. Indeed, let $P, Q \in R_T$, and $\alpha, \beta \in \mathbb{R}$. Since T is quasihomogeneous, $T(\beta Q) = \beta Q$, which implies

$$T(\alpha P + \beta Q) = T(\alpha P + T(\beta Q)) = T(\alpha P) + T(\beta Q) = \alpha P + \beta Q,$$

because T is quasiadditive. This finishes the proof. $\square$

The next corollary provides special cases of best approximation operator assuming additional properties on φ.

Corollary 3.7. *Let $\varphi \in \mathcal{F}$ be a differentiable strictly convex function with $\varphi'(0) = 0$. Let $T : L^\varphi \to L^\varphi$ be a single-valued operator such that T is quasiadditive, quasihomogeneous, and satisfies*

$$\int_B \varphi'(|f - Tf|)sgn(f - Tf)T(f)dm \geq 0, \quad \text{for all} \quad f \in L^\varphi.$$

Then R_T is a subspace of L^φ, and $Tf = \mu_\varphi(f/R_T)$.

Proof. Let $f \in L^\varphi$ be such that $Tf \neq 0$ and we consider $r : (0, 1] \to \mathbb{R}$, the strictly convex function defined by

$$r(t) = \frac{1}{t} \int_B (\varphi(|f - Tf + tTf|) - \varphi(|f - Tf|))dm.$$

By hypothesis we have $\int_B \varphi(|f|)dm - \int_B \varphi(|f - Tf|)dm = r(1) \geq r(t)$, $t \in (0, 1]$. So, ([11], pp. 16-17) implies that

$$\begin{aligned}\int_B \varphi(|f|)dm - \int_B \varphi(|f - Tf|)dm &\geq \lim_{t\to 0^+} r(t)\\ &\geq \int_B \varphi'(|f - Tf|)sgn(f - Tf)T(f)dm \geq 0,\end{aligned}$$

i.e. T is φ-expectation invariant. Since $\mu_\varphi(f/R_T)$ is a singleton for all $f \in L^\varphi$, Theorem 3.6 completes the proof. $\square$

We point out that the extended best polynomial approximation operator given in [7], defined on $L^{p-1}(B)$, satisfies the hypothesis of Corollary 3.7.

In the following we consider additional properties on T, which allow us to obtain specific subspaces R_T.

Definition 3.8. *Let S be a subspace of L^φ. We say that S is measurable if there exists a sub σ-algebra of $\mathcal{A}$, say $\mathcal{L}$, such that S is the class of all $\mathcal{L}$-measurable functions in L^φ, i.e., $S = L^\varphi(B, \mathcal{L}, m)$. The subspace S is called bounded if $S \subset \overline{S \cap L^\infty(B)}^\varphi$.*

Lemma 3.9. *Let $S \subset L^\varphi$ be a subspace. If S is measurable, then $\overline{S}^\varphi = S$.*

Proof. By hypothesis $S = L^\varphi(B, \mathcal{L}, m)$ for some σ-subalgebra $\mathcal{L}$ of $\mathcal{A}$. If $g \in \overline{S}^\varphi$, then there exists a sequence $\{g_k\}_{k\in\mathbb{N}} \subset S$ such that

$$\lim_{k\to\infty} d_\varphi(g_k, g) = 0. \tag{2}$$

Hence, there exists a subsequence of $\{g_k\}_{k\in\mathbb{N}}$, which is denoted in the same way, such that $\lim_{k\to\infty} g_k = g$ a.e. on B. So, g is $\mathcal{L}$-measurable function. In addition, from (2) we have $g \in L^\varphi$ and consequently $g \in S$. This completes the proof. □

Example 3.10. *The subspace S given in Example 3.2 (b) is bounded and measurable. Indeed, it is easy to check that $S = L^\varphi(B, \mathcal{L}, m) \subset L^\infty(B)$, where $\mathcal{L}$ is the sub σ-algebra $\left\{\emptyset, \left[0, \frac{1}{2}\right), \left[\frac{1}{2}, 1\right], B\right\}$.*

Lemma 3.11. *Let $\varphi \in \mathcal{F}$ and $T : L^\varphi \to L^\varphi$ be a single-valued operator such that T is quasiadditive, quasialgebraic and φ-closed. Then the set $\mathcal{L} = \{A \subset B : T(\chi_A) = \chi_A\}$ is a sub σ-algebra of $\mathcal{A}$.*

Proof. Clearly $B \in \mathcal{L}$ by $T(1) = 1$. Let $A_1, A_2 \in \mathcal{L}$. As $\chi_{A_1\cap A_2} = \chi_{A_1}\chi_{A_2} = T(\chi_{A_1})T(\chi_{A_2})$, then

$$T(\chi_{A_1\cap A_2}) = T(T(\chi_{A_1})T(\chi_{A_2})) = T(\chi_{A_1})T(\chi_{A_1}) = \chi_{A_1}\chi_{A_2} = \chi_{A_1\cap A_2}. \tag{3}$$

According to (3), we have $\chi_{A_1\setminus A_2} = \chi_{A_1} - \chi_{A_1\cap A_2} = \chi_{A_1} - T(\chi_{A_1\cap A_2})$, hence (1) implies

$$T(\chi_{A_1\setminus A_2}) = T(\chi_{A_1} - T(\chi_{A_1\cap A_2})) = T(\chi_{A_1}) - T(\chi_{A_1\cap A_2}) = \chi_{A_1} - \chi_{A_1\cap A_2} = \chi_{A_1\setminus A_2}. \tag{4}$$

By (4) we get

$$\begin{aligned} T(\chi_{A_1\cup A_2}) &= T(\chi_{A_1} + \chi_{A_2\setminus A_1}) = T(\chi_{A_1} + T(\chi_{A_2\setminus A_1})) = T(\chi_{A_1}) + T(\chi_{A_1\setminus A_2}) \\ &= \chi_{A_1} + \chi_{A_1\setminus A_2} = \chi_{A_1\cup A_2}. \end{aligned} \tag{5}$$

Now, let $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{L}$. From (5) we obtain that $g_N := \chi_{\bigcup_{k=1}^N A_k} \in \mathcal{L}$, $N \in \mathbb{N}$, and $\{g_N\}_{N\in\mathbb{N}}$ is a nondecreasing sequence such that $\lim_{N\to\infty} d_\varphi(g_N, g) = 0$ where $g = \chi_{\bigcup_{k=1}^\infty A_k}$. Since $T(g_N) = g_N$, $N \in \mathbb{N}$, and T is φ-closed, it follows that $T(g) = g$. The proof is complete. □

Theorem 3.12. *Let $\varphi \in \mathcal{F}$ and let $T : L^\varphi \to L^\varphi$ be a single-valued operator. Assume that $\mu_\varphi(f/R_T)$ is a singleton for all $f \in L^\varphi$. If T is quasiadditive, quasialgebraic, φ-closed, φ-expectation invariant, and φ-bounded, then R_T is a bounded measurable subspace of L^φ, and $Tf = \mu_\varphi(f/R_T)$ for all $f \in L^\varphi$.*

Proof. Since T is quasialgebraic, then it is quasihomogeneous and $T0 = 0$. By Lemma 3.3, we have

$$R_T = \{f \in L^\varphi \ : \ Tf = f\}.$$

Let $\mathcal{L}$ be the sub σ-algebra given in Lemma 3.11, and let $A_1, A_2 \in \mathcal{L}$, $\alpha_1, \alpha_2 \in \mathbb{R}$. By hypothesis we have

$$\begin{aligned} T(\alpha_1\chi_{A_1} + \alpha_2\chi_{A_2}) &= T(\alpha_1\chi_{A_1} + \alpha_2 T(\chi_{A_2})) = T(\alpha_1\chi_{A_1} + T(\alpha_2 T(\chi_{A_2}))) \\ &= T(\alpha_1\chi_{A_1}) + T(\alpha_2 T(\chi_{A_2})) = T(\alpha_1 T(\chi_{A_1})) + T(\alpha_2 T(\chi_{A_2})) \\ &= \alpha_1 T(\chi_{A_1}) + \alpha_2 T(\chi_{A_2}) = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2}. \end{aligned} \tag{6}$$

Therefore $\alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} \in R_T$. So, linear combinations of characteristic functions of sets of $\mathcal{L}$ are in R_T. The facts that the totality of such functions is dense in $L^\varphi(B, \mathcal{L}, m)$ respect to d_φ, and T is φ-closed, imply $T(f) = f$ for all $f \in L^\varphi(B, \mathcal{L}, m)$. Hence

$$L^\varphi(B, \mathcal{L}, m) \subset R_T. \tag{7}$$

We claim that

$$R_T \cap L^\infty(B) \subset L^\varphi(B, \mathcal{L}, m). \tag{8}$$

Indeed, let $g \in R_T \cap L^\infty(B)$. Then there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq g \leq \beta$. Hence, $\int_B \varphi(|g|)dm \leq \varphi(|\max\{|\alpha|, |\beta|\}) m(B) < \infty$ and so $g \in L^\varphi$. Set $I = [\alpha, \beta]$ and let $\mathcal{M}$ be the Borel σ-algebra on I. Now, we will prove that if $D \in \mathcal{M}$, then $g^{-1}(D) \in \mathcal{L}$, and in consequence $g \in L^\varphi(B, \mathcal{L}, m)$.

We consider the Lebesgue Stieltjes measure $\mu_g : \mathcal{M} \to [0, m(B)]$ given by

$$\mu_g(C) = m(g^{-1}(C)) \quad \text{for} \quad C \in \mathcal{M}. \tag{9}$$

Since $(I,\mathcal{M},\mu_g)$ is a finite measure space, from (9), ([4], Prop.7.2, p.80) and ([4], Prop.1.8, p.43), we have

$$\int_I \varphi(|h|)d\mu_g = \int_B \varphi(|h\circ g|)dm \text{ whenever } h: I\to\mathbb{R} \text{ is a } \mathcal{M}-\text{measurable function.} \tag{10}$$

Let $D\in\mathcal{M}$ and $k\in\mathbb{N}$. As $\varphi(0^+)=0$, there is $\epsilon_k>0$ such that

$$\varphi(\epsilon_k) < \frac{1}{2k\Lambda}\min\left\{\frac{1}{\varphi(1)},\frac{1}{m(I)}\right\}.$$

Since, μ_g is a regular measure, there are an open set U and a closed set F such that $F\subset D\subset U\subset I$ and $\mu_g(U\setminus F)<\varphi(\epsilon_k)$. By Urysohn's Lemma, there exists a continuous function $f: I\to[0,1]$ such that $f\,|_F=1$ and $f\,|_{(I\setminus U)}=0$. So,

$$\int_I \varphi(|f-\chi_D|)d\mu_g = \int_{U\setminus F}\varphi(|f-\chi_D|)d\mu_g \le \varphi(1)\mu_g(U\setminus F) < \frac{1}{2k\Lambda}. \tag{11}$$

On the other hand, by Weierstrass's Theorem then there exists a polynomial P_k on I such that $|P_k(x)-f(x)|<\epsilon_k$ for all $x\in I$, and so, $\int_I\varphi(|P_k-f|)d\mu_g \le \varphi(\epsilon_k)m(I) < \frac{1}{2k\Lambda}$. Therefore, (10) and (11) imply that

$$\begin{aligned}\int_B \varphi(|P_k(g)-\chi_{g^{-1}(D)}|)dm &= \int_B\varphi(|(P_k-\chi_D)\circ g|)dm = \int_I\varphi(|P_k-\chi_D|)d\mu_g\\ &\le \Lambda\int_I\varphi(|P_k-f|)d\mu_g + \Lambda\int_I\varphi(|f-\chi_D|)d\mu_g < \frac{1}{k},\end{aligned} \tag{12}$$

and consequently $\lim_{k\to\infty} d_\varphi(P_k(g),\chi_{g^{-1}(D)})=0$.

Since $g^2=TgTg=T(TgTg)=T(g^2)$, then by induction we see that $g^n=T(g^n)$, $n\in\mathbb{N}$, i.e., $g^n\in R_T$, $n\in\mathbb{N}$. By Theorem 3.6 it follows that R_T is a subspace of L^φ. Hence, $P_k(g)\in R_T$, $k\in\mathbb{N}$, and so $\lim_{k\to\infty} d_\varphi(T(P_k(g)),\chi_{g^{-1}(D)})=0$. As T is φ-closed, we have $T\chi_{g^{-1}(D)}=\chi_{g^{-1}(D)}$ and therefore $g^{-1}(D)\in\mathcal{L}$.

Now, from (7), (8) and Lemmas 3.3 and 3.9, we get $\overline{R_T\cap L^\infty(B)}^{\varphi}\subset L^\varphi(B,\mathcal{L},m)\subset R_T$. Since T is φ-bounded, we have $R_T=L^\varphi(B,\mathcal{L},m)$.

Finally, Theorem 3.6 completes the proof. □

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