Strong uniqueness and alternation theorems for relative Chebyshev centers[☆]

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Abstract

In this paper, we give a strong uniqueness characterization theorem for the Chebyshev center of a set of infinitely many functions relative to a finite-dimensional linear space on a compact Hausdorff space. Additionally, we derive an alternation theorem for Chebyshev centers relative to a weak Chebyshev space on any compact set of the real line. Furthermore, we show an intrinsic characterization of those linear spaces where an alternation theorem holds.

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1. Introduction

Let $\mathbf{U}$ be a finite-dimensional linear space of continuous function on a compact Hausdorff space X. The well-known Haar Unicity Theorem [10] establishes that each continuous function f on X has a unique element in $\mathbf{U}$ that is a best Chebyshev approximation to f from $\mathbf{U}$ if and only if $\mathbf{U}$ is spanned by a Haar system. For such linear spaces, the Chebyshev Theorem (on alternation) holds [23]. Jones and Karlovitz [13] showed that weak Chebyshev systems of continuous functions on a compact interval $[a, b]$ of $\mathbb{R}$ completely characterize those linear spaces $\mathbf{U}$ for which each continuous function has at least one best Chebyshev approximation for which the alternation theorem holds. It is natural to wonder whether such a result remains valid if we replace $[a, b]$ with an arbitrary compact subset of the real line. Deutsch, Nürnberger, and Singer [6] have provided an affirmative answer to this question.

On the other hand, we have simultaneous Chebyshev approximation, which deals with the best Chebyshev approximation to sets of functions. This subject has a long history [9, 12, 24] and can be viewed as a special

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case of vector-valued approximations [21]. Tanimoto [27] derived a necessary and sufficient condition for a function to be a best simultaneous Chebyshev approximation to a finite set of functions. A best simultaneous Chebyshev approximation to a certain set Δ from a subset $\mathbf{U}$ in a normed linear space is also known as a Chebyshev center of Δ relative to $\mathbf{U}$, or the relative Chebyshev center of Δ in $\mathbf{U}$. This topic was defined by Garkavi [8] in the sixties, and lines of research regarding the existence, uniqueness, and characterization of relative Chebyshev centers have been developed since then (see for instance [2, 5, 18, 26]). Additionally, a recent survey about relative Chebyshev centers can be found in [1].

The general simultaneous approximation problem can sometimes be reduced to a problem involving the approximation of two functions. In particular, problems related to alternation in the context of simultaneous approximation to two functions from Haar spaces and generalizations have been investigated by several authors. Amir and Ziegler [3] established a Chebyshev alternation theorem for the best simultaneous Chebyshev approximation to two functions from n-unisolvent families of continuous functions on the compact interval $[0, 1]$. Fernández and Soriano [7] gave another alternation theorem for the best simultaneous approximation to two functions from a Haar space of continuous function on a compact interval $[a, b]$, in the particular case when the approximation criteria is given by a monotone norm with the commutative property. Later, Tanimoto [28] showed an alternation theorem and a strong uniqueness result for the best simultaneous Chebyshev approximation to a set of infinitely many functions from a Haar space on a compact interval of $\mathbb{R}$.

The study of strong uniqueness for the relative Chebyshev center in a subset has also been under research for a long time. Prus and Smarzewski [22] proved that in a uniformly convex space with a modulus of convexity of power type $q \geq 2$, the relative Chebyshev center of a bounded set in a closed convex set is strongly unique of order q. Furthermore, by using the one-sided Gateaux derivative of the deviation of a point to a set in a locally convex space, Laurent and Pai [15] gave a strong uniqueness characterization theorem for the relative center of a bounded set in a linear space involving a seminorm. In a similar context, results on the relative center in generalized set-suns were given by Luo, Li, and He [17]. Further, Li [16] showed that the relative Chebyshev center of a bounded set in an so called RS-set in a real Banach space is strongly unique. Moreover, in case where the set of relative centers is not a singleton, Pai and Indira [19] showed an intrinsic characterization of the finite-dimensional linear space, called a property (Li). This characterization yields Hausdorff strong uniqueness of relative Chebyshev centers.

The main goal of our paper is to give a strong uniqueness characterization theorem for the Chebyshev center of a set of infinitely many functions relative to a finite-dimensional linear space on a compact Hausdorff space. Moreover, we derive an alternation theorem for Chebyshev centers relative to a weak Chebyshev space on compact sets on $\mathbb{R}$. This paper extends previous pieces of work in two directions. First, we extend best Chebyshev approximation results to the more general setting of relative Chebyshev center. Secondly, known results of weak Chebyshev systems on compact intervals are extended to any compact set of the real line.

It is well known that the Haar systems (or Chebyshev systems) play an important role in many parts of analysis, as well as in probability and statistics. However, the weak Chebyshev systems are weaker forms of Haar systems, capable of encompassing splines. Since classes of spline functions possess many nice structural properties as well as excellent approximation powers, a myriad of applications in the numerical solution of a variety of problems in applied mathematics may be found.

The remainder of this paper is organized as follows. Notations, definitions, and essential results needed are given in Section 2, followed by Section 3 which presents some results about weak Chebyshev systems. In Section 4, we show characterization, uniqueness, and sign changes of Chebyshev centers of a set of infinitely many functions. An alternation theorem from a finite-dimensional linear space spanned by a weak Chebyshev system is studied in Section 5. Additionally, we derive an intrinsic characterization of those linear spaces where an alternation theorem holds.

2. Preliminaries

Throughout, the symbol X will designate any compact set of a Hausdorff space such that they contain at least $n+1$ distinct points, where n is a given natural integer. Further, X^* will stand for the convex hull of X.

We denote by $\mathcal{C}(X)$ the space of continuous real-valued functions on X and we will use $\|\cdot\|_X$ to denote the uniform norm on $\mathcal{C}(X)$.

If $Y \subset X$ and $\mathbf{U}$ is any n-dimensional linear space of $\mathcal{C}(X)$, we will write $\dim(\mathbf{U})$ for the dimension of $\mathbf{U}$ and designate by $\mathbf{U}_Y := \{\underline{u}_{_Y} : u \in \mathbf{U}\}$, where $\underline{u}_{_Y}$ denotes the restriction of u to Y.

When the Hausdorff space is $\mathbb{R}$, we will write M instead of X, and use $\mathbf{U} = U$. In this case, M^* is a compact interval of $\mathbb{R}$.

Definition 2.1. *Let Δ be a set of uniformly bounded functions in $\mathcal{C}(X)$ and let $\mathbf{U}$ be an n-dimensional linear space of $\mathcal{C}(X)$. We say that an element u^* in $\mathbf{U}$ is a relative Chebyshev center (r.c.c.) of Δ in $\mathbf{U}$ on X if*

$$\sup_{f\in\Delta} \|f-u^*\|_X \le \sup_{f\in\Delta} \|f-u\|_X, \quad \textit{for all} \quad u \in \mathbf{U}. \tag{2.1}$$

By $Z_{X,\mathbf{U}}(\Delta)$, we denote the set of all $u^* \in \mathbf{U}$ fulfilling (2.1). The number

$$r_{X,\mathbf{U}}(\Delta) := \inf_{u\in\mathbf{U}} \sup_{f\in\Delta} \|f-u\|_X$$

is called the relative Chebyshev radius of Δ in $\mathbf{U}$. In particular, if $\Delta = \{f_1, f_2\}$, then we say that u^* in $\mathbf{U}$ is a r.c.c. to f_1 and f_2 in $\mathbf{U}$ on X (or a best simultaneous Chebyshev approximation to f_1 and f_2 from $\mathbf{U}$ on X) and write $Z_{X,\mathbf{U}}(f_1,f_2)$ and $r_{X,\mathbf{U}}(f_1,f_2)$ instead of $Z_{X,\mathbf{U}}(\Delta)$ and $r_{X,\mathbf{U}}(\Delta)$, respectively.

It is well known that for $f_1, f_2 \in \mathcal{C}(X)$, $Z_{X,\mathbf{U}}(f_1, f_2)$ is a non-empty set [9, Lemma 2.2], although it is not necessarily unitary. Note that

$$\frac{1}{2}\|f_1 - f_2\|_X \leq \inf_{u \in \mathbf{U}} \max\{\|f_1 - u\|_X, \|f_2 - u\|_X\} = r_{X,\mathbf{U}}(f_1, f_2). \tag{2.2}$$

In particular, if $f = f_1 = f_2$, then $Z_{X,\mathbf{U}}(f) := Z_{X,\mathbf{U}}(f_1, f_2)$ is the set of best Chebyshev approximations to f from $\mathbf{U}$ on X.

Let Δ be a set of uniformly bounded functions in $\mathcal{C}(X)$ and set

$$f_-(x) := \inf_{f \in \Delta} f(x) \quad \text{and} \quad f_+(x) := \sup_{f \in \Delta} f(x), \quad x \in X. \tag{2.3}$$

It is clear that f_- and f_+ are not necessarily continuous. For example if $\Delta = \{f_n\}$ where $f_n(x) = 1 - x^n$ on $[0, 1]$, then $f_+(x) = \operatorname{sgn}(1 - x)$.

Definition 2.2. *Let Δ be a set of uniformly bounded functions in $\mathcal{C}(X)$. The set Δ is said to be complete if $f_- \in \mathcal{C}(X)$ and $f_+ \in \mathcal{C}(X)$. Moreover, if for each $x \in X$, there exist $g, h \in \Delta$ such that $f_-(x) = g(x)$ and $f_+(x) = h(x)$, we say that the set Δ is totally complete.*

We observe that if Δ is a finite set, it is clearly complete.

An immediate consequence of [28, Lemma 1] is the following characterization theorem.

Theorem 2.3. *Let Δ be a complete set of uniformly bounded functions in $\mathcal{C}(X)$ and let $\mathbf{U} \subset \mathcal{C}(X)$ be an n-dimensional linear space. Then $u^* \in Z_{X,\mathbf{U}}(\Delta)$ if and only if $u^* \in Z_{X,\mathbf{U}}(f_-, f_+)$. Further,*

$$r_{X,\mathbf{U}}(\Delta) = \sup_{f \in \Delta} \|f - u^*\|_X = \max\{\|f_- - u^*\|_X, \|f_+ - u^*\|_X\}. \tag{2.4}$$

From (2.2) and (2.4) it follows that

$$\frac{1}{2}\|f_- - f_+\|_X \leq r_{X,\mathbf{U}}(\Delta).$$

On the other hand, it is well known that the number $\inf_{u \in \mathcal{C}(X)} \sup_{f \in \Delta} \|f - u\|_X$ is called the Chebyshev radius of Δ. When $\frac{f_- + f_+}{2} \in \mathcal{C}(X)$, it is easy to see that the Chebyshev radius of Δ coincide with $\frac{1}{2}\|f_- - f_+\|_X$.

We also have the following characterization result for relative Chebyshev centers that is proved by Tanimoto in [28, Theorem 2].

Theorem 2.4. *Let Δ be a totally complete set of uniformly bounded functions in $\mathcal{C}(X)$ and let $\mathbf{U} \subset \mathcal{C}(X)$ be an n-dimensional linear space. Assume $u^* \in U$. The following statements are equivalent:*

(a) $u^* \in Z_{X,\mathbf{U}}(\Delta)$,

(b) there exist $\lambda_1, \ldots, \lambda_s > 0$, *s distinct elements* $z_1, z_2, \ldots, z_s$ *in* X, *and s functions* $h_1, \ldots, h_s \in \Delta$, *with* $1 \le s \le n+1$, *such that*

(b1) $|h_i(z_i) - u^*(z_i)| = r_{X,\mathbf{U}}(\Delta)$, $1 \le i \le s$,

(b2) $\sum_{i=1}^{s} \lambda_i (h_i(z_i) - u^*(z_i)) u(z_i) = 0$ *for all* $u \in \mathbf{U}$.

Let $\{g_1, \ldots, g_\ell\}$ be a set of continuous functions defined on X, and let $z_1, z_2, \ldots, z_\ell$ be distinct points in X. We denote the Gramian of the Gram matrix $(g_j(z_i))_{i,j=1}^{\ell}$ by

$$V \begin{pmatrix} g_1, \ldots, g_{\ell-1}, g_\ell \\ z_1, \ldots, z_{\ell-1}, z_\ell \end{pmatrix} := \det \left((g_j(z_i))_{i,j=1}^{\ell} \right).$$

Definition 2.5. *A set* $\{u_1, \ldots, u_n\}$ *of functions in* $\mathcal{C}(X)$ *is called an Haar system (H-system) on* X *if*

$$V \begin{pmatrix} u_1, \ldots, u_{n-1}, u_n \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix} \neq 0, \quad \text{for every choice of } n \text{ distinct elements } z_1, \ldots, z_n \text{ in } X.$$

An n-dimensional linear space $\mathbf{U}$ *of* $\mathcal{C}(X)$ *is called an Haar space (H-space) on* X *if there exists a basis* $\{u_1, \ldots, u_n\}$ *for* $\mathbf{U}$ *such that it is an H-system on* X.

An H-space on M^* is generally called a Chebyshev space (T-space). For an n-dimensional linear space U of $\mathcal{C}(M^*)$, it is well known that U is a T-space on M^* if and only if there exists a basis $\{u_1, \ldots, u_n\}$ for U such that

$$V \begin{pmatrix} u_1, \ldots, u_{n-1}, u_n \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix} > 0, \quad \text{for all } z_1 < z_2 < \ldots < z_n \text{ in } M^*. \tag{2.5}$$

When we have a Haar space, the characterization result Theorem 2.4 can be further strengthened. Precisely, Tanimoto [28, Theorem 3] proved the following alternation theorem for the relative Chebyshev center.

Theorem 2.6. *Let* Δ *be a totally complete set of uniformly bounded functions in* $\mathcal{C}(M^*)$ *and let* $U \subset \mathcal{C}(M^*)$ *be an n-dimensional H-space on* M^*. *Assume* $u^* \in U$. *Then the following statements are equivalent:*

(a) $u^* \in Z_{M^*,U}(\Delta)$,

(b) there exist $n+1$ *elements* $z_1 < z_2 < \ldots < z_{n+1}$ *in* M^*, *and* $n+1$ *functions* $h_1, \ldots, h_{n+1} \in \Delta$ *such that*

(b1) $|h_i(z_i) - u^*(z_i)| = r_{M^*,U}(\Delta)$, $1 \le i \le n+1$,

(b2) $h_{i+1}(z_{i+1}) - u^*(z_{i+1}) = (-1)^i (h_i(z_i) - u^*(z_i))$, $1 \le i \le n$.

Definition 2.7. *An n-dimensional H-space* U *of* $\mathcal{C}(M)$ *is called a strong Haar space (SH-space) on* M *if there exists a basis* $\{u_1, \ldots, u_n\}$ *for* U *such that (2.5) holds on* M. *Such a basis is said to be a SH-system on* M.

We observe that if $\{u_1, \ldots, u_n\}$ is a basis for an n-dimensional linear space of $\mathcal{C}(M)$ satisfying (2.5) on M, then clearly $\{u_1, \ldots, u_n\}$ is a Haar system on M. However, in contrast to the case when $X = M^*$, not every Haar system on M satisfies (2.5) on M.

Example 2.8. *Let $u_1(x) = \sin(x)$ and $u_2(x) = -\cos(x)$. It is easy to verify that $\{u_1, u_2\}$ is an H-system on $M = \left[\frac{\pi}{4}, \frac{3\pi}{8}\right] \cup \left[\frac{3\pi}{2}, \frac{7\pi}{4}\right]$, but it is not a SH-system on M.*

For a vector of real numbers $\omega = (\omega_1, \ldots, \omega_r)$, let $S^-(\omega)$ be the number of sign changes in the sequence $\omega_1, \ldots, \omega_r$, where zeros are ignored. Note that $S^-(\omega) = 0$ when $r = 1$.

Let $f \in \mathcal{C}(M)$. We recall that

$$S_M^-(f) := \sup_{\ell}\{S^-((f(z_1), \ldots, f(z_\ell))) : z_1 < \ldots < z_\ell \text{ in } M\}$$

counts the number of strong sign changes of f on M [25, Definition 2.11]. Of course, if f is either non-negative or non-positive on M, then we set $S_M^-(f) = 0$.

Definition 2.9. *[25, Definition 2.35] A set $\{u_1, \ldots, u_n\}$ of linearly independent functions in $\mathcal{C}(M)$ is called a weak Chebyshev system (WT-system) on M if*

$$V\begin{pmatrix} u_1, \ldots, u_{n-1}, u_n \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix} \geq 0, \quad \textit{for all } z_1 < z_2 < \ldots < z_n \textit{ in } M.$$

An n-dimensional linear space U of $\mathcal{C}(M)$ is called a weak Chebyshev space (WT-space) on M if there exists a basis $\{u_1, \ldots, u_n\}$ for U such that it is a WT-system on M.

Example 2.10. *Let $u_1(x) = 1$ and $u_2(x) = \max\{0, x\}$. Clearly, $\{u_1, u_2\}$ is a WT-system on $M = [-2, -1] \cup [1, 2]$, but it is not an H-system on M.*

Example 2.11. *Let $u_1(x) = \sin(x)$ and $u_2(x) = -\cos(x)$. It is easy to verify that $\{u_1, u_2\}$ is a WT-system on $[0, \pi]$, but it is not a WT-system on $[0, 2\pi]$.*

Good references for a description and more examples of all these systems are [11, 25].

The following two theorems provide characterizations of WT-systems and WT-spaces.

Theorem 2.12. *[25, Theorem 2.39] Let $\{u_1, \ldots, u_n\}$ be a set of linearly independent functions in $\mathcal{C}(M)$ and $U = \operatorname{span}\{u_1, \ldots, u_n\}$. If $\{u_1, \ldots, u_n\}$ is a WT-system on M, then*

$$S_M^-(u) \leq n - 1, \quad \textit{for all} \quad u \in U \setminus \{0\}. \tag{2.6}$$

Conversely, if (2.6) holds, then either $\{u_1, \ldots, u_{n-1}, u_n\}$, or $\{u_1, \ldots, u_{n-1}, -u_n\}$ is a WT-system on M.

Below, we present a result by Deutsch, Nürnberger, and Singer that completely characterizes those finite-dimensional linear spaces on a compact set of $\mathbb{R}$ for which an alternation theorem holds.

Theorem 2.13. *[6, Theorem 4.1] Let $U \subset \mathcal{C}(M)$ be an n-dimensional linear space. A necessary and sufficient conditions that U is an n-dimensional WT-space on M is that for each $f \in \mathcal{C}(M)$ there exist at least one $u^* \in Z_{M,U}(f)$, $\sigma \in \{-1, 1\}$, and $n+1$ elements $z_1 < z_2 < \ldots < z_{n+1}$ in M such that*

$$f(z_i) - u^*(z_i) = \sigma(-1)^{n+1-i} r_{M,U}(f), \quad 1 \leq i \leq n+1.$$

The theorem that follows is a 'smoothing' procedure proved in [20, Proposition 6, p.199] for functions defined on any compact interval of the real line.

Theorem 2.14. *[20, Proposition 6, p.199] Let $\{u_1, \ldots, u_n\} \subset \mathcal{C}(M^*)$ be a WT-system on M^*. Then, for every $\epsilon > 0$, there exists a T-system $\{u_1^\epsilon, \ldots, u_n^\epsilon\}$ on M^* such that each u_j^ϵ uniformly converges as $\epsilon \to 0$ to u_j on M^*, $1 \leq j \leq n$.*

3. Extension of weak Chebyshev systems

It is well known that if $V \subset \mathcal{C}(M^*)$ is an ℓ-dimensional WT-space, then $U = V_M \subset \mathcal{C}(M)$ is an n-dimensional WT-space where $n \leq \ell$ (see for instance [25, Theorem 2.40]). Now suppose that $U \subset \mathcal{C}(M)$ is an n-dimensional WT-space on M. We wonder if we could find an n-dimensional WT-space V on M^* such that $U = V_M$. The next part will show that this question has an affirmative answer. Lastly, we will prove a 'smoothing' procedure.

We begin by giving the following notations and an auxiliary lemma.

If $M \subsetneq M^*$, then $M^* \setminus M$ is a non-empty open set on M^*. So, there exists a countable collection $\{(a_i, b_i) : i \in L\}$, $L \subset \mathbb{N}$, of pairwise disjoint open intervals of $\mathbb{R}$ such that

$$M^* \setminus M = \bigcup_{i \in L} (a_i, b_i).$$

Definition 3.1. *Under the assumptions above, for $g \in \mathcal{C}(M)$, we will use $T_M(g)$ to denote the continuous extension of g to M^* given by*

$$T_M(g)(x) = \begin{cases} g(x) & \text{if} \quad x \in M, \\ \frac{g(b_i)-g(a_i)}{b_i - a_i}(x - a_i) + g(a_i) & \text{if} \quad x \in (a_i, b_i), \quad i \in L. \end{cases}$$

We will use $T_M(g) = g$ provided that $M = M^*$.

Lemma 3.2. *The operator $T_M : \mathcal{C}(M) \to \mathcal{C}(M^*)$ is a linear map such that $\|T_M(g)\|_{M^*} = \|g\|_M$ and $S^-_{M^*}(T_M(g)) = S^-_M(g)$ for all $g \in \mathcal{C}(M)$.*

Proof. Clearly, T_M is a linear map. Let $g \in \mathcal{C}(M)$. If $x \in (a_i, b_i)$, $i \in L$, then $|T_M(g)(x)| \le \max\{|g(a_i)|, |g(b_i)|\} \le \|g\|_M$ since $a_i, b_i \in M$. Therefore, $\|T_M(g)\|_{M^*} = \|g\|_M$.

On the other hand, as

$$\begin{aligned} &\{S^-((g(z_1), \ldots, g(z_\ell))) : z_1 < \ldots < z_\ell \text{ in } M\} \\ &= \{S^-((T_M(g)(z_1), \ldots, T_M(g)(z_\ell))) : z_1 < \ldots < z_\ell \text{ in } M\} \\ &\subset \{S^-((T_M(g)(z_1), \ldots, T_M(g)(z_\ell))) : z_1 < \ldots < z_\ell \text{ in } M^*\}, \end{aligned}$$

it follows that $S_M^-(g) \le S_{M^*}^-(T_M(g))$. Assume $\ell = S_M^-(g) < S_{M^*}^-(T_M(g))$. Hence, there exist $z_1 < \ldots < z_{\ell+2}$ in M^* such that

$$S^-((T_M(g)(z_1), \ldots, T_M(g)(z_{\ell+1}), T_M(g)(z_{\ell+2}))) = \ell + 1, \tag{3.1}$$

that is, $T_M(g)(z_i)T_M(g)(z_{i+1}) < 0$, $1 \le i \le \ell + 1$. If $z_1 < \ldots < z_{\ell+2}$ in M, then

$$S^-((g(z_1), \ldots, g(z_{\ell+1}), g(z_{\ell+2}))) = \ell + 1, \tag{3.2}$$

which is a contradiction. So, there is $1 \le k \le \ell + 2$ that verifies $z_k \notin M$. Set

$$m_1 = \min\{k : 1 \le k \le \ell + 2, z_k \notin M\}$$

and let $i \in L$ be such that $z_{m_1} \in (a_i, b_i)$.

Suppose that $m_1 = \ell + 2$. We observe that $z_{\ell+1} \le a_i$ since $z_{\ell+1} \in M$. As $T_M(g)(z_{m_1}) \ne 0$, then either $g(a_i) \ne g(b_i)$, or $g(a_i) = g(b_i) \ne 0$. So, we can find $z \in \{a_i, b_i\} \setminus \{z_{\ell+1}\}$ satisfying $T_M(g)(z_{m_1})g(z) > 0$ and $z_1 < \ldots < z_{\ell+1} < z$ in M. In fact, we consider three cases. In the first case, we assume $T_M(g)(z_{m_1})g(a_i) > 0$. As $T_M(g)(z_{m_1})g(z_{\ell+1}) = T_M(g)(z_{m_1})T_M(g)(z_{\ell+1}) < 0$, we have $z_{\ell+1} < a_i$, and take $z = a_i$. In the second case, we suppose that $T_M(g)(z_{m_1})g(a_i) < 0$. Since $T_M(g)(a_i) = g(a_i)$, $T_M(g)$ is a linear function on the interval $[a_i, b_i]$ and $z_{m_1} \in (a_i, b_i)$, we get $T_M(g)(z_{m_1})g(b_i) > 0$. Thus, we put $z = b_i$. In the third case, $g(a_i) = 0$, and therefore, $g(b_i) \ne 0$. Consequently, $T_M(g)(z_{m_1})g(b_i) > 0$ and we consider $z = b_i$. It follows that $\operatorname{sgn}(T_M(g)(z_{\ell+1}))\operatorname{sgn}(T_M(g)(z)) < 0$, and hence

$$S^-((g(z_1), \ldots, g(z_{\ell+1}), g(z))) = S^-((T_M(g)(z_1), \ldots, T_M(g)(z_{\ell+1}), T_M(g)(z))) = \ell + 1,$$

which is impossible.

Now assume $m_1 < \ell + 2$. Then either $z_{m_1+1} < b_i$ or $z_{m_1+1} \ge b_i$. We consider each case separately.

Case (I): $z_{m_1+1} < b_i$. Since $T_M(g)(z_{m_1})T_M(g)(z_{m_1+1}) < 0$, then $g(a_i) \ne g(b_i)$, $\operatorname{sgn}(T_M(g)(z_{m_1})) = \operatorname{sgn}(g(a_i))$, and $\operatorname{sgn}(T_M(g)(z_{m_1+1})) = \operatorname{sgn}(g(b_i))$. Therefore, $z_{m_1-1} < a_i < z_{m_1}$ if $m_1 > 1$, and $z_{m_1+1} < b_i < z_{m_1+2}$ if $m_1 + 1 < \ell + 2$. Now, we replace z_{m_1} with a_i and z_{m_1+1} with b_i.

Case (II): $z_{m_1+1} \ge b_i$. Proceeding as before, we can take $z \in \{a_i, b_i\} \setminus \{z_{m_1+1}\}$ such that $T_M(g)(z_{m_1})g(z) > 0$.

We replace z_{m_1} with z.

In either case, we have a new set of points $z_1 < \ldots < z_{\ell+2}$ in M^* for which $m_1 < \min\{k : 1 \le k \le \ell+2, z_k \notin M\}$ and (3.1) holds. Repeating the steps above, we can find $z_1 < \ldots < z_{\ell+2}$ in M that satisfies (3.2), which is another contradiction. This completes the proof. $\square$

Theorem 3.3. *Let $\{u_1, \ldots, u_n\}$ be a set of linearly independent functions in $\mathcal{C}(M)$. Then, $\{u_1, \ldots, u_n\}$ is a WT-system on M if and only if $\{T_M(u_1), \ldots, T_M(u_n)\}$ is a WT-system on M^*.*

Proof. Assume that $\{u_1, \ldots, u_n\}$ is a WT-system on M. If $M = M^*$, it is obvious. Now assume $M \subsetneq M^*$. Since $\{u_1, \ldots, u_n\}$ is a set of linearly independent functions, we have

$$V\begin{pmatrix} u_1, \ldots, u_{n-1}, u_n \\ y_1, \ldots, y_{n-1}, y_n \end{pmatrix} \neq 0, \quad \text{for some } y_1 < y_2 < \ldots < y_n \text{ in } M. \tag{3.3}$$

Therefore,

$$V\begin{pmatrix} T_M(u_1), \ldots, T_M(u_{n-1}), T_M(u_n) \\ y_1, \ldots, y_{n-1}, y_n \end{pmatrix} \neq 0, \quad \text{for some } y_1 < y_2 < \ldots < y_n \text{ in } M^*,$$

and so $\{T_M(u_1), \ldots, T_M(u_n)\}$ is a set of linearly independent functions. According to Lemma 3.2 and Theorem 2.12, we have

$$S^-_{M^*}\left(\sum_{j=1}^n c_j T_M(u_j)\right) = S^-_{M^*}\left(T_M\left(\sum_{j=1}^n c_j u_j\right)\right) = S^-_M\left(\sum_{j=1}^n c_j u_j\right) \le n-1,$$

for any real $c_1, \ldots, c_n$ not all 0. From Theorem 2.12 it may be concluded that either $\{T_M(u_1), \ldots, T_M(u_n)\}$, or $\{T_M(u_1), \ldots, -T_M(u_n)\}$ is a WT-system on M^*. Suppose that $\{T_M(u_1), \ldots, -T_M(u_n)\}$ is a WT-system on M^* and let $z_1 < z_2 < \ldots < z_n$ in M. Since $\{u_1, \ldots, u_n\}$ is a WT-system on M, we obtain

$$\begin{aligned} 0 \le V&\begin{pmatrix} T_M(u_1), \ldots, T_M(u_{n-1}), -T_M(u_n) \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix} = V\begin{pmatrix} u_1, \ldots, u_{n-1}, -u_n \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix} \\ &= -V\begin{pmatrix} u_1, \ldots, u_{n-1}, u_n \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix} \le 0. \end{aligned}$$

As $z_1 < z_2 < \ldots < z_n$ in M are arbitrary, we deduce that $V\begin{pmatrix} u_1, \ldots, u_{n-1}, u_n \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix} = 0$ for all $z_1 < z_2 < \ldots < z_n$ in M, which is contrary to (3.3).

Conversely, we suppose that $\{T_M(u_1), \ldots, T_M(u_n)\}$ is a WT-system on M^*. It follows easily that $\{u_1, \ldots, u_n\}$ is a WT-system on M. This completes the proof. $\square$

The example below shows that Theorem 3.3 does not remain valid for H-systems in general.

Example 3.4. *Let $M = [a, b] \cup [c, d]$ be such that $b < c$, and we consider that $u_1(x) = 1$, $u_2(x) = x$, and $u_3(x) = x^2$. It is well known that $\{u_1, u_2, u_3\}$ is an H-system on M. On the other hand, since $(bc\,u_1 - (b+c)u_2 + u_3)(b) = (bc\,u_1 - (b+c)u_2 + u_3)(c) = 0$, we have $T_M(bc\,u_1 - (b+c)u_2 + u_3) = 0$ on $[b, c]$ by Definition 3.1. Thus, Lemma 3.2 becomes $bc\,T_M(u_1) - (b+c)T_M(u_2) + T_M(u_3) = 0$ on $[b, c]$, and so $\{T_M(u_1), T_M(u_2), T_M(u_3)\}$ is not a T-system on $M^* = [a, d]$.*

The theorem that follows shows that Theorem 2.14 is also valid for functions defined on any compact set of the real line.

Theorem 3.5. *Let $\{u_1, \ldots, u_n\} \subset \mathcal{C}(M)$ be a WT-system on M. Then, for every $\epsilon > 0$, there exists a SH-system $\{u_1^\epsilon, \ldots, u_n^\epsilon\}$ on M such that each u_j^ϵ uniformly converges as $\epsilon \to 0$ to u_j on M, $1 \le j \le n$.*

Proof. If $M = M^*$, the result is obvious by Theorem 2.14. Now assume $M \subsetneq M^*$. Theorem 3.3 implies that $\{T_M(u_1), \ldots, T_M(u_n)\}$ is a WT-system on M^*. According to Theorem 2.14, we have that for every $\epsilon > 0$, there exists a T-system $\{v_1^\epsilon, \ldots, v_n^\epsilon\}$ on M^* such that each v_j^ϵ converges uniformly as $\epsilon \to 0$ to $T_M(u_j)$ on M^*, $1 \le j \le n$. We write $u_j^\epsilon = \underline{v_j^\epsilon}_M \in \mathcal{C}(M)$, and therefore u_j^ϵ converges uniformly as $\epsilon \to 0$ to u_j on M, $1 \le j \le n$. Finally, as $\{u_1^\epsilon, \ldots, u_n^\epsilon\}$ is a SH-system on M, the proof is complete. □

4. Characterization and uniqueness of r.c.c. in finite-dimensional linear spaces

In this section, we provide some results concerning the characterization and uniqueness of r.c.c. in a finite-dimensional linear space.

Some ways of turning the simultaneous Chebyshev approximation problem into an approximation involving a single function can be found in [5, p.51]. The result below shows another different way.

Lemma 4.1. *Let $f_1, f_2 \in \mathcal{C}(X)$ and let $\mathbf{U}$ be an n-dimensional linear space of $\mathcal{C}(X)$. Let Y be the compact set $[0, 1] \times X$ in the product of topological spaces, and let $F : Y \to \mathbb{R}$ be the continuous function given by $F(t, x) = tf_1(x) + (1-t)f_2(x)$. For $u \in \mathbf{U}$, we consider $L_u : Y \to \mathbb{R}$ defined by $L_u(t, x) = u(x)$ and let $\mathbf{V} = \{L_u : u \in \mathbf{U}\}$ be the n-dimensional linear space of $\mathcal{C}(Y)$. Then $u^* \in Z_{X,\mathbf{U}}(f_1, f_2)$ if and only if L_{u^*} is a best Chebyshev approximation to F from $\mathbf{V}$ on Y. Further,*

$$r_{X,\mathbf{U}}(f_1, f_2) = \inf_{u \in \mathbf{U}} \|F - L_u\|_Y, \quad u \in \mathbf{U}. \tag{4.1}$$

Proof. A straightforward computation shows that

$$\max\{\|f_1 - u\|_X, \|f_2 - u\|_X\} = \|F - L_u\|_Y, \quad u \in \mathbf{U}. \tag{4.2}$$

Hence, the lemma immediately follows. □

For a set $\{u_1, \ldots, u_n\}$ of linearly independent functions in $\mathcal{C}(X)$ and n distinct elements $y_1, \ldots, y_n$ in X, for the sake of simplicity, we write

$$D(y_1, \ldots, y_n) := V \begin{pmatrix} u_1, \ldots, u_{n-1}, u_n \\ y_1, \ldots, y_{n-1}, y_n \end{pmatrix}.$$

Lemma 4.2. *Under the hypothesis of Lemma 4.1 and s distinct elements $(t_1, z_1), (t_2, z_2), \ldots, (t_s, z_s)$ in Y with $1 \leq m \leq s \leq 2n$, let $\mathbf{V}_{\{(t_1,z_1),\ldots,(t_s,z_s)\}} = \left\{ \underline{L_u}_{\{(t_1,z_1),\ldots,(t_s,z_s)\}} : u \in \mathbf{U} \right\}$. We consider the following statements:*

(a) $\dim(\mathbf{V}_{\{(t_1,z_1),\ldots,(t_s,z_s)\}}) = m$,

(b) $\dim(\mathbf{U}_{\{z_1,\ldots,z_s\}}) = m$,

(c) there exist a basis $\{u_1, \ldots, u_n\}$ of $\mathbf{U}$ and a set $\{z_{i_1}, \ldots, z_{i_n}\}$ of distinct elements of $\{z_1, \ldots, z_s\}$ such that $D(z_{i_1}, \ldots, z_{i_n}) > 0$.

Then, (a) and (b) are equivalent. Moreover, if $m = n$, then (b) $\Longrightarrow$ (c).

Proof. $(a) \Longleftrightarrow (b)$ Set $P = \{(t_1, z_1), \ldots, (t_s, z_s)\}$ and $Q = \{z_1, \ldots, z_s\}$. Note that $\dim(\mathbf{V}_P) = m$ if and only if there exists $\{u_1, \ldots, u_m\} \subset \mathbf{U}$ such that $\left\{ \underline{L_{u_1}}_P, \ldots, \underline{L_{u_m}}_P \right\}$ is a basis of $\mathbf{V}_P$. This condition is equivalent to the matrix

$$\mathbf{A}_m = \begin{pmatrix} u_1(z_1) & u_2(z_1) & \cdots & u_m(z_1) \\ u_1(z_2) & u_2(z_2) & \cdots & u_m(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(z_s) & u_2(z_s) & \cdots & u_m(z_s) \end{pmatrix}$$

having a full column rank equal to m, or equivalent to $\left\{ \underline{u_1}_Q, \ldots, \underline{u_m}_Q \right\}$ which is a basis of $\mathbf{U}_Q$, that is, $\dim(\mathbf{U}_Q) = m$.

$(b) \Longrightarrow (c)$ Finally, we suppose that $m = n$. Then, $\dim(\mathbf{U}_Q) = n$ is equivalent to the matrix $\mathbf{A}_n$ having a column rank equal to n, that is, $\mathbf{A}_n$ has a row rank equal to n, This condition is equivalent to the existence of $z_{i_1}, \ldots, z_{i_n}$ such that vectors $\mathbf{v}_r = (u_1(z_{i_r}), u_2(z_{i_r}), \ldots, u_n(z_{i_r}))$, $1 \leq r \leq n$, are linearly independent, or equivalent to $D(z_{i_1}, \ldots, z_{i_n}) \neq 0$. This implies that $\{u_1, \ldots, u_n\}$ is a basis of $\mathbf{U}$ and $z_{i_1}, \ldots, z_{i_m}$ are distinct elements. Finally, if $D(z_{i_1}, \ldots, z_{i_n}) < 0$, then we can take the basis $\{u_1, \ldots, -u_n\}$ of $\mathbf{U}$ and the proof is complete. □

Let $(E, \|\cdot\|)$ be a normed space and $S \subset E$. Assume that for a given $f \in E$, there exists a best approximation s^* to f from S on E, that is, $\|f - s^*\| \leq \|f - s\|$ for all $s \in S$. We recall that s^* is strongly unique if there exists $\gamma > 0$ depending on f such that

$$\|f - s^*\| + \gamma\|s^* - s\| \le \|f - s\|, \quad \text{for all} \quad s \in S.$$

This leads to the following generalization of the above concept of strong uniqueness (see for instance [16, p.43]).

Definition 4.3. *Let Δ be a set of uniformly bounded functions in $\mathcal{C}(X)$ and let $\mathbf{U} \subset \mathcal{C}(X)$ be an n-dimensional linear space. We say that $u^* \in Z_{X,\mathbf{U}(\Delta)}$ is strongly unique if there exists $\gamma > 0$ depending on Δ such that*

$$\sup_{f\in\Delta} \|f - u^*\|_X + \gamma\|u^* - u\|_X \le \sup_{f\in\Delta} \|f - u\|_X, \quad \textit{for all} \quad u \in \mathbf{U}.$$

Remark 4.4. *The following equality has been proved in [4, p.130] for real-valued function defined on an interval. However, it is clear that it is also valid for functions defined on any compact set.*

$$\sup_{f\in\Delta} \|f - u\|_X = \left\| \left| \frac{f_- + f_+}{2} - u \right| + \left| \frac{f_- - f_+}{2} \right| \right\|_X, \quad u \in \mathbf{U},$$

provided that $f_-, f_+ \in \mathcal{C}(X)$. On the other hand, since

$$\left| \frac{f_-(x) + f_+(x)}{2} - u(x) \right| + \left| \frac{f_-(x) - f_+(x)}{2} \right| = \max\{|f_-(x) - u(x)|, |f_+(x) - u(x)|\}, \quad x \in X, \quad u \in \mathbf{U},$$

we deduce that

$$\left\| \left| \frac{f_- + f_+}{2} - u \right| + \left| \frac{f_- - f_+}{2} \right| \right\|_X = \max\{\|f_- - u\|_X, \|f_+ - u\|_X\}, \quad u \in \mathbf{U}.$$

Below, we show that Theorem 2.4 can be further refined. Further, this theorem extends [14, Theorem 2.4] for relative Chebyshev center problems.

Theorem 4.5. *Let Δ be a complete set of uniformly bounded functions in $\mathcal{C}(X)$ and let $\mathbf{U} \subset \mathcal{C}(X)$ be an n-dimensional linear space. Assume that $u^* \in \mathbf{U}$. The following statements are equivalent:*

(a) $u^* \in Z_{X,\mathbf{U}}(\Delta)$ is strongly unique,

(b) there exist $\lambda_1, \ldots, \lambda_s > 0$, s distinct elements $z_1, z_2, \ldots, z_s$ in X, and s functions $h_1, \ldots, h_s \in \{f_-, f_+\}$, with $n + 1 \le s \le 2n$, such that

- *(b1) $|h_i(z_i) - u^*(z_i)| = r_{X,\mathbf{U}}(\Delta)$, $1 \le i \le s$,*
- *(b2) $\sum_{i=1}^{s} \lambda_i(h_i(z_i) - u^*(z_i))u(z_i) = 0$ for all $u \in \mathbf{U}$,*
- *(b3) $\dim(\mathbf{U}_{\{z_1,\ldots,z_s\}}) = n$.*

Proof. If $r_{X,\mathbf{U}}(\Delta) = 0$, the equivalence is obvious since $f_- = f_+ = u^*$ by Theorem 2.3. Now we assume that $r_{X,\mathbf{U}}(\Delta) > 0$. According to Remark 4.4 and (4.2), we have that

$$\sup_{f\in\Delta} \|f - u\|_X = \|F - L_u\|_Y, \quad u \in \mathbf{U},$$

where $F(t, x) = tf_-(x)+(1-t)f_+(x)$. We can deduce that $u^* \in Z_{X,\mathbf{U}}(\Delta)$ is strongly unique if and only if L_{u^*} is the strongly unique best Chebyshev approximation to F from $\mathbf{V}$ on Y, or equivalently, by [14, Theorem 2.4], that there exist $\mu_1, \dots, \mu_\ell > 0$, ℓ distinct elements $(t_1, z_1), \dots, (t_\ell, z_\ell)$ in Y, with $n + 1 \le \ell \le 2n$, such that

(i) $|F(t_i, z_i) - L_{u^*}(t_i, z_i)| = \inf_{u\in\mathbf{U}} \|F - L_u\|_Y$, $1 \le i \le \ell$,

(ii) $\sum_{i=1}^{\ell} \mu_i(F(t_i, z_i) - L_{u^*}(t_i, z_i))L_u(t_i, z_i) = 0$ for all $u \in \mathbf{U}$,

(iii) $\dim(\mathbf{V}_{\{(t_1,z_1),\dots,(t_\ell,z_\ell)\}}) = n$.

Note that $n = \dim(\mathbf{V}_{\{(t_1,z_1),\dots,(t_\ell,z_\ell)\}}) = \dim(\mathbf{U}_{\{z_1,\dots,z_\ell\}})$ by Lemma 4.2, and $L_u(t_i, z_i) = u(z_i)$, $u \in \mathbf{U}$, $1 \le i \le \ell$. Further, from (2.4) and (4.1), we have that

$$r_{X,\mathbf{U}}(\Delta) = r_{X,\mathbf{U}}(f_-, f_+) = \inf_{u\in\mathbf{U}} \|F - L_u\|_Y. \tag{4.3}$$

$(a) \implies (b)$ Let $1 \le i \le \ell$. Since $0 \le t_i \le 1$, from (i) and (4.3), we deduce that

$$\begin{aligned} r_{X,\mathbf{U}}(\Delta) &= |t_i(f_-(z_i) - u^*(z_i)) + (1 - t_i)(f_+(z_i) - u^*(z_i))| \\ &\le t_i|f_-(z_i) - u^*(z_i)| + (1 - t_i)|f_+(z_i) - u^*(z_i)| \\ &\le t_i\|f_- - u^*\|_X + (1 - t_i)\|f_+ - u^*\|_X \le r_{X,\mathbf{U}}(f_-, f_+) = r_{X,\mathbf{U}}(\Delta). \end{aligned} \tag{4.4}$$

Then, $t_i|f_-(z_i) - u^*(z_i)| = t_i\|f_- - u^*\|_X$, $(1 - t_i)|f_+(z_i) - u^*(z_i)| = (1 - t_i)\|f_+ - u^*\|_X$ and

$$t_i(1 - t_i)(f_+(z_i) - u^*(z_i))(f_-(z_i) - u^*(z_i)) \ge 0. \tag{4.5}$$

If $t_i = 0$, then $F(t_i, z_i) = f_+(z_i)$. On the other hand, $F(t_i, z_i) = f_-(z_i)$ provided $t_i = 1$. Now, suppose $t_i \notin \{0, 1\}$. We conclude from (4.4) that $\|f_- - u^*\|_X = \|f_+ - u^*\|_X$, hence that $|f_-(z_i) - u^*(z_i)| = |f_+(z_i) - u^*(z_i)|$, and finally that $f_-(z_i) = f_+(z_i) = F(t_i, z_i)$ by (4.5). Next, we deduce that

(1) $|h_i(z_i) - u^*(z_i)| = r_{X,\mathbf{U}}(\Delta)$, $1 \le i \le \ell$,

(2) $\sum_{i=1}^{\ell} \mu_i(h_i(z_i) - u^*(z_i))u(z_i) = 0$ for all $u \in \mathbf{U}$,

(3) $\dim(\mathbf{U}_{\{z_1,\dots,z_s\}}) = n$,

By Lemma 4.2, there exist a basis $\{u_1, \ldots, u_n\}$ of $\mathbf{U}$ and a set $Q = \{z_{i_1}, \ldots, z_{i_n}\}$ of distinct elements of $\{z_1, \ldots, z_\ell\}$ such that $D(z_{i_1}, \ldots, z_{i_n}) > 0$.

We claim that

$$\text{the cardinal of } \{z_1, \ldots, z_\ell\} \setminus Q \text{ is at least one.} \tag{4.6}$$

Indeed, on the contrary, we suppose that $\{z_1, \ldots, z_\ell\} \setminus Q = \emptyset$. For $1 \le r \le n$ we write

$$I_r = \{z_k : 1 \le k \le \ell \text{ and } z_k = z_{i_r}\} \quad \text{and} \quad \beta_r = \sum_{z_k \in I_r} \mu_k.$$

It is easy to verify that $\beta_r > 0$, $1 \le r \le n$, and $\{I_r : 1 \le r \le n\}$ is a collection of pairwise disjoint sets such that $\bigcup_{r=1}^{n} I_r = \{z_1, \ldots, z_\ell\}$. Hence, (2) shows that

$$\sum_{r=1}^{n} \beta_r (h_{i_r}(z_{i_r}) - u^*(z_{i_r})) u_j(z_{i_r}) = 0, \quad 1 \le j \le n.$$

Since $D(z_{i_1}, \ldots, z_{i_n}) > 0$, it follows that $h_{i_r}(z_{i_r}) - u^*(z_{i_r}) = 0$, $1 \le r \le n$. Therefore, from (1) we have $r_{X,\mathbf{U}}(\Delta) = 0$, which a contradiction.

It only remains to prove that

$$\ell \text{ can be chosen so that } z_1, \ldots, z_\ell \text{ are distinct elements and } n + 1 \le \ell. \tag{4.7}$$

Let $s \in \mathbb{N}$ be the cardinal of $\{z_1, \ldots, z_\ell\}$. From (4.6), we deduce that $n+1 \le s \le \ell$. Let $P = \{z_{i_1}, \ldots, z_{i_s}\} \subset \{z_1, \ldots, z_\ell\}$ such that P has s distinct elements. By the same argument above, for $1 \le k \le s$ we write

$$I_k = \{z_t : z_t \in \{z_1, \ldots, z_\ell\} \text{ and } z_t = z_{i_k}\} \quad \text{and} \quad \lambda_k = \sum_{z_t \in I_k} \mu_t > 0.$$

Therefore, according to (2), we obtain

$$\sum_{k=1}^{s} \lambda_k (h_{i_k}(z_{i_k}) - u^*(z_{i_k})) u(z_{i_k}) = 0, \quad \text{for all} \quad u \in \mathbf{U}.$$

So, we can see that there exist $\lambda_1, \ldots, \lambda_s > 0$, s distinct elements $z_1, z_2, \ldots, z_s$ in X, and s functions $h_1, \ldots, h_s \in \{f_-, f_+\}$, with $n + 1 \le s \le 2n$, such that (b1) and (b2) hold.

$(b) \implies (a)$ Let $t_i = 1$ if $h_i = f_-$, otherwise $t_i = 0$. Hence, $F(t_i, z_i) = h_i$, $1 \le i \le s$. Therefore, from (b1)-(b2), it follows that (i) and (ii) hold, and so the proof of (a) is complete. □

Remark 4.6. *By definition of an n-dimensional H-space $\mathbf{U}$ on X, we always have $\dim(\mathbf{U}_{\{z_1,\ldots,z_{n+1}\}}) = n$ for every choice of $n+1$ distinct points $z_1, \ldots, z_{n+1}$ in X.*

When we have an H-space, the characterization Theorem 4.5 can be further strengthened. Additionally, we derive a property of r.c.c. in terms of sign changes. We recall that this result does not differ much from

Theorem 2.6. However, it is proved for functions defined on any compact set without the assumption of Δ to be totally complete.

Theorem 4.7. *Let Δ be a complete set of uniformly bounded functions in $\mathcal{C}(X)$ and let $\mathbf{U} \subset \mathcal{C}(X)$ be an n-dimensional H-space spanned by an H-system $\{u_1, \ldots, u_n\}$ on X. Assume that $u^* \in \mathbf{U}$. The following statements are equivalent:*

(a) $u^* \in Z_{X,\mathbf{U}}(\Delta)$,

(b) there exist $\sigma \in \{-1, 1\}$, $n+1$ *distinct elements* $z_1, z_2, \ldots, z_{n+1}$ *in* X, *and* $n+1$ *functions* $h_1, \ldots, h_{n+1} \in \{f_-, f_+\}$ *such that*

$$h_i(z_i) - u^*(z_i) = \sigma(-1)^{n+1-i}\mathrm{sgn}\left(\frac{D(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n+1})}{D(z_1, \ldots, z_n)}\right) r_{X,\mathbf{U}}(\Delta), \quad 1 \le i \le n+1. \tag{4.8}$$

Proof. If $r_{X,\mathbf{U}}(\Delta) = 0$, the equivalence is obvious since $f_- = f_+ = u^*$. Now, we assume $r_{X,\mathbf{U}}(\Delta) > 0$. Following the same argument as Theorem 4.5 with [14, Theorem 2.3] instead of [14, Theorem 2.4], we deduce that $u^* \in Z_{X,\mathbf{U}}(\Delta)$ if and only if there exist $\lambda_1, \ldots, \lambda_s > 0$, s distinct elements $z_1, z_2, \ldots, z_s$ in X, and s functions $h_1, \ldots, h_s \in \{f_-, f_+\}$, with $1 \le s \le n+1$, such that

(i) $|h_i(z_i) - u^*(z_i)| = r_{X,\mathbf{U}}(\Delta)$, $1 \le i \le s$,

(ii) $\sum_{i=1}^{s} \lambda_i (h_i(z_i) - u^*(z_i)) u(z_i) = 0$ for all $u \in \mathbf{U}$.

We claim that $s = n+1$.

In fact, on the contrary, we assume $s \le n$. Since $\{u_1, \ldots, u_n\}$ is an H-system on X, there exists $u \in \mathbf{U}$ such that $u(z_i) = h_i(z_i) - u^*(z_i)$, $1 \le i \le s$. Hence, (ii) leads to

$$0 = \sum_{i=1}^{s} \lambda_i (h_i(z_i) - u^*(z_i))^2 = r_{X,\mathbf{U}}(\Delta)^2 \sum_{i=1}^{s} \lambda_i,$$

which is a contradiction.

Note that $D(z_1, \ldots, z_n) \neq 0$ and the condition (ii) is equivalent to

$$\sum_{i=1}^{n} \frac{\lambda_i (h_i(z_i) - u^*(z_i))}{-\lambda_{n+1}(h_{n+1}(z_{n+1}) - u^*(z_{n+1}))} u_j(z_i) = u_j(z_{n+1}), \quad 1 \le j \le n. \tag{4.9}$$

(a) $\Longrightarrow$ (b) From (4.9) and by using Cramer's Rule, it follows that

$$\begin{aligned} \frac{\lambda_i (h_i(z_i) - u^*(z_i))}{-\lambda_{n+1}(h_{n+1}(z_{n+1}) - u^*(z_{n+1}))} &= \frac{D(z_1, \ldots, z_{i-1}, z_{n+1}, z_{i+1}, \ldots, z_n)}{D(z_1, \ldots, z_n)} \\ &= (-1)^{n-i} \frac{D(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n+1})}{D(z_1, \ldots, z_n)}, \end{aligned} \tag{4.10}$$

$1 \le i \le n$. Since (i) holds, by taking modulus to both sides of (4.10) we get

$$\frac{\lambda_i}{\lambda_{n+1}} = \operatorname{sgn}\left(\frac{D(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_{n+1})}{D(z_1,\ldots,z_n)}\right)\frac{D(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_{n+1})}{D(z_1,\ldots,z_n)},$$

$1 \le i \le n$. This proves (b) with $\sigma = \operatorname{sgn}(h_{n+1}(z_{n+1}) - u^*(z_{n+1}))$.

(b) $\Longrightarrow$ (a) Clearly, (i) is obvious. The proof is completed by showing that (ii) holds, or equivalently, that (4.9) holds. Let $\lambda_{n+1} = 1$ and $\lambda_i = \left|\frac{D(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_{n+1})}{D(z_1,\ldots,z_n)}\right|$, $1 \le i \le n$. Since $\lambda_i > 0$, according to (b) we obtain

$$\frac{\lambda_i(h_i(z_i) - u^*(z_i))}{-\lambda_{n+1}(h_{n+1}(z_{n+1}) - u^*(z_{n+1}))} = \frac{D(z_1,\ldots,z_{i-1},z_{n+1},z_{i+1},\ldots,z_n)}{D(z_1,\ldots,z_n)},$$

$1 \le i \le n$. Hence, $\left(\frac{\lambda_1(h_1(z_1)-u^*(z_1))}{-\lambda_{n+1}(h_{n+1}(z_{n+1})-u^*(z_{n+1}))},\ldots,\frac{\lambda_n(h_n(z_n)-u^*(z_n))}{-\lambda_{n+1}(h_{n+1}(z_{n+1})-u^*(z_{n+1}))}\right)$ is the unique solution of the system of n linear equations given by

$$\sum_{i=1}^{n} c_i u_j(z_i) = u_j(z_{n+1}), \quad 1 \le j \le n.$$

So (4.9) is true, and the proof is complete. □

Remark 4.8. *It is evident that if we have M instead of X, then the distinct elements $z_1,\ldots,z_n$ in M given in Theorem 4.7 can be chosen so that $z_1 < z_2 < \ldots < z_{n+1}$.*

Corollary 4.9. *Let Δ be a complete set of uniformly bounded functions in $\mathcal{C}(X)$ and let $\mathbf{U} \subset \mathcal{C}(X)$ be an n-dimensional H-space on X spanned by an H-system $\{u_1,\ldots,u_n\}$. Assume that $\frac{1}{2}\|f_- - f_+\|_X < r_{X,\mathbf{U}}(\Delta)$ and $u^* \in Z_{X,\mathbf{U}}(\Delta)$. Then, there exist $\sigma \in \{-1,1\}$, $n+1$ distinct elements $z_1, z_2, \ldots, z_{n+1}$ in X, and $n+1$ functions $h_1,\ldots,\ h_{n+1} \in \{f_-, f_+\}$ such that item (b) of Theorem 4.7 is satisfied and*

$$\operatorname{sgn}\left(\left(\frac{f_- + f_+}{2} - u^*\right)(z_i)\right) = \sigma(-1)^{n+1-i}\operatorname{sgn}\left(\frac{D(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_{n+1})}{D(z_1,\ldots,z_n)}\right), \tag{4.11}$$

$1 \le i \le n+1$.

Proof. By Theorem 4.7, there exist $\sigma \in \{-1,1\}$, $n+1$ distinct elements $z_1, z_2, \ldots, z_{n+1}$ in X, and $n+1$ functions $h_1,\ldots,\ h_{n+1} \in \{f_-, f_+\}$ such that item (b) of Theorem 4.7 is satisfied.

If $h_i = f_-$, by hypothesis we have $\frac{1}{2}\,|f_-(z_i) - f_+(z_i)| \le \frac{1}{2}\|f_- - f_+\|_X < r_{X,\mathbf{U}}(\Delta) = |f_-(z_i) - u^*(z_i)|$, and so

$$\operatorname{sgn}\left(h_i(z_i) - u^*(z_i)\right) = \operatorname{sgn}\left(f_-(z_i) - u^*(z_i)\right) = \operatorname{sgn}\left(\left(\frac{f_- + f_+}{2}\right)(z_i) - u^*(z_i)\right), \tag{4.12}$$

where the last equality uses the fact that $\operatorname{sgn}(b) = \operatorname{sgn}(b-a)$ provide $|a| < |b|$. Similarly, if $h_i = f_+$ we obtain $\frac{1}{2}\,|f_+(z_i) - f_-(z_i)| < |f_+(z_i) - u^*(z_i)|$, and hence (4.12) is also true. Therefore, (4.11) holds by (4.12), and (b) of Theorem 4.7. □

The following is an example where $\frac{1}{2}\|f_- - f_+\|_X < r_{X,\mathbf{U}}(\Delta)$.

Example 4.10. *Let $X = \left[\frac{1}{4}, 1\right]$, $u(x) = x$, and let Δ be any complete set of uniformly bounded functions in $\mathcal{C}(X)$ such that $f_-(x) = 0$ and $f_+(x) = 1$. Assume that $\mathbf{U} \subset \mathcal{C}(X)$ is the 1-dimensional linear space spanned by $\{u\}$. It is easy to check that $\{u\}$ is an H-system on X, $\frac{1}{2}\|f_- - f_+\|_X = \frac{1}{2}$, and $r_{X,\mathbf{U}}(\Delta) = \frac{4}{5}$. Furthermore, $u^*(x) = \frac{4}{5}x$, $h_1 = f_+$, $h_2 = f_-$, $z_1 = \frac{1}{4}$, and $z_2 = 1$ verify (4.8) and (4.11) with $\sigma = 1$.*

Corollary 4.11. *Let $\mathbf{U} \subset \mathcal{C}(X)$ be an n-dimensional H-space on X. Then, for each complete set Δ of uniformly bounded functions in $\mathcal{C}(X)$, $u^* \in Z_{X,\mathbf{U}}(\Delta)$ is strongly unique.*

Proof. It follows immediately from the proof of Theorem 4.7, Remark 4.6, and Theorem 4.5. □

Corollary 4.12. *Let $\mathbf{U} \subset \mathcal{C}(X)$ be an n-dimensional linear space. The following statements are equivalent:*

(a) $\mathbf{U}$ is an n-dimensional H-space on X,

(b) For each complete set Δ of uniformly bounded functions in $\mathcal{C}(X)$, there exists a unique $u^* \in Z_{X,\mathbf{U}}(\Delta)$.

Proof. $(a) \Longrightarrow (b)$ It is obvious from Corollary 4.11.
$(b) \Longrightarrow (a)$ Follows from [14, Theorem 4.1] by taking $\Delta = \{f\}$. □

In the specific case where U is a finite-dimensional SH-space, Theorem 4.7 can be further refined into a final and more geometric form. Moreover, it shows that Theorem 2.6 is also valid for functions defined on any compact set of the real line.

Theorem 4.13. *Let Δ be a complete set of uniformly bounded functions in $\mathcal{C}(M)$ and let $U \subset \mathcal{C}(M)$ be an n-dimensional SH-space on M. Assume that $u^* \in U$. The following statements are equivalent:*

(a) $u^* \in Z_{M,U}(\Delta)$,

(b) there exist $\sigma \in \{-1, 1\}$, $n+1$ elements $z_1 < z_2 < \ldots < z_{n+1}$ in M, and $n+1$ functions $h_1, \ldots, h_{n+1} \in \{f_-, f_+\}$ such that

$$h_i(z_i) - u^*(z_i) = \sigma(-1)^{n+1-i} r_{M,U}(\Delta), \quad 1 \leq i \leq n+1. \tag{4.13}$$

Proof. Assume that U is spanned by a SH-system on M, $\{u_1, \ldots, u_n\}$. If $r_{M,U}(\Delta) = 0$, the equivalence is obvious. Now, we assume $r_{M,U}(\Delta) > 0$. Since U is an H-space, according to Theorem 4.7 and Remark 4.8, we have $u^* \in Z_{M,U}(f_-, f_+)$ if and only if there exist $\sigma \in \{-1, 1\}$, $n+1$ distinct elements $z_1 < z_2 < \ldots < z_{n+1}$ in M, and $n+1$ functions $h_1, \ldots, h_{n+1} \in \{f_-, f_+\}$, such that

$$h_i(z_i) - u^*(z_i) = \sigma(-1)^{n+1-i} \operatorname{sgn}\left(\frac{D(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n+1})}{D(z_1, \ldots, z_n)}\right) r_{M,U}(\Delta), \quad 1 \leq i \leq n+1.$$

As $z_1 < z_2 < \ldots < z_n < z_{n+1}$ and $\{u_1, \ldots, u_n\}$ is a SH-system on M, we obtain

$$D(z_1, \ldots, z_n) > 0 \quad \text{and} \quad D(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n+1}) > 0, \quad 1 \leq i \leq n+1. \tag{4.14}$$

This completes the proof. $\square$

Corollary 4.14. *Let Δ be a complete set of uniformly bounded functions in $\mathcal{C}(M)$, and let $U \subset \mathcal{C}(M)$ be an n-dimensional SH-space on M. Assume that $\frac{1}{2}\|f_- - f_+\|_M < r_{M,U}(\Delta)$ and $u^* \in Z_{M,U}(\Delta)$. Then, there exist $\sigma \in \{-1,1\}$, $n+1$ elements $z_1 < z_2 < \ldots < z_{n+1}$ in M, and $n+1$ functions $h_1, \ldots, h_{n+1} \in \{f_-, f_+\}$ such that item (b) of Theorem 4.13 is satisfied and*

$$\left(\frac{f_- + f_+}{2} - u^*\right)(z_i)\left(\frac{f_- + f_+}{2} - u^*\right)(z_{i+1}) < 0, \quad 1 \leq i \leq n. \tag{4.15}$$

Proof. Assume that U is spanned by a SH-system on M, $\{u_1, \ldots, u_n\}$. Then, it is an immediate consequence of (4.11), (4.14), and Theorem 4.13. $\square$

5. An alternation theorem for r.c.c. from WT-systems

In this section, we derive an alternation theorem for Chebyshev centers relative to WT-spaces on compact sets on $\mathbb{R}$. First, we establish a relationship between the convergence of a net of n-dimensional linear spaces and the convergence of any net of Cheyshev centers relative to such linear spaces.

Theorem 5.1. *Let $\{u_1^\epsilon, \ldots, u_n^\epsilon\}$ be sets of linearly independent functions in $\mathcal{C}(X)$, $0 \leq \epsilon < 1$. Assume Δ is a complete set of uniformly bounded functions in $\mathcal{C}(X)$, $\mathbf{U}_\epsilon = \operatorname{span}\{u_1^\epsilon, \ldots, u_n^\epsilon\}$ and each u_j^ϵ converges uniformly as $\epsilon \to 0$ to u_j^0 on X, $1 \leq j \leq n$. Then, for any choice of $u_\epsilon^* \in Z_{X,\mathbf{U}_\epsilon}(\Delta)$, there exist a subnet of $\{u_\epsilon^*\}$, which we denote, for convenience, using the same index ϵ, and $u^* \in \mathbf{U}_0$ such that*

(a) $\lim_{\epsilon \to 0} \|u_\epsilon^* - u^*\|_X = 0$,

(b) $\lim_{\epsilon \to 0} r_{X,\mathbf{U}_\epsilon}(\Delta) = r_{X,\mathbf{U}_0}(\Delta)$,

(c) $u^* \in Z_{X,\mathbf{U}_0}(\Delta)$.

Proof. Let $u_\epsilon^* \in Z_{X,\mathbf{U}_\epsilon}(\Delta)$, $0 < \epsilon < 1$, and suppose that $u_\epsilon^* = \sum_{j=1}^{n} c_j^\epsilon u_j^\epsilon$. We observe that

$$\|u_\epsilon^*\|_X \leq \|f_+ - u_\epsilon^*\|_X + \|f_+\|_X \leq r_{X,\mathbf{U}_\epsilon}(\Delta) + \|f_+\|_X \leq 2 \sup_{f \in \Delta} \|f\|_X < \infty.$$

Since $\{u_1^0, \ldots, u_n^0\}$ is a set of linearly independent functions in $\mathcal{C}(X)$, there are $z_1 < \ldots < z_n$ in X for which

$$V\begin{pmatrix} u_1^0, \ldots, u_{n-1}^0, u_n^0 \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix} \neq 0.$$

By hypothesis, there exist $0 < \epsilon_0 < 1$ and $\kappa_0 > 0$ such that $\|u_\epsilon^*\|_X \leq \kappa_0$ and $|u_j^\epsilon(z_i)| \leq \kappa_0$ for all $0 < \epsilon < \epsilon_0$ and $1 \leq i, j \leq n$. As the determinant is a continuous function, we also deduce that

$$\lim_{\epsilon \to 0}\left|V\begin{pmatrix} u_1^\epsilon, \ldots, u_{n-1}^\epsilon, u_n^\epsilon \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix}\right| = \left|V\begin{pmatrix} u_1^0, \ldots, u_{n-1}^0, u_n^0 \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix}\right| > 0.$$

So, there exist $0 < \epsilon_1 < 1$ and $\kappa_1 > 0$ for which $\kappa_1 < \left| V \begin{pmatrix} u_1^\epsilon, \ldots, u_{n-1}^\epsilon, u_n^\epsilon \\ z_1, \ldots, z_{n-1}, z_n \end{pmatrix} \right|$ for all $0 < \epsilon < \epsilon_1$. By using Cramer's Rule we have

$$|c_j^\epsilon| \leq \frac{\kappa_0^n n!}{\kappa_1}, \quad 0 < \epsilon < \min\{\epsilon_0, \epsilon_1\}, \quad 1 \leq j \leq n. \tag{5.1}$$

Now, we can find a subnet, which we denote other time with the same index ϵ, such that c_j^ϵ converges as $\epsilon \to 0$ to $c_j \in \mathbb{R}$, $1 \leq j \leq n$. Set $u^* = \sum_{j=1}^{n} c_i u_j^0 \in \mathbf{U}_0$. We observe that

$$\|u_\epsilon^* - u^*\|_X \leq \frac{\kappa_0^n n!}{\kappa_1} \sum_{j=1}^{n} \|u_j^\epsilon - u_j^0\|_X + \sum_{j=1}^{n} |c_j^\epsilon - c_j| \|u_j^0\|_X, \quad 0 < \epsilon < \min\{\epsilon_0, \epsilon_1\}.$$

Thus, (a) holds.

Since $\sup_{f\in\Delta} \|f - u^*\|_X \leq \sup_{f\in\Delta} \|f - u_\epsilon^*\|_X + \|u_\epsilon^* - u^*\|_X = r_{X,\mathbf{U}_\epsilon}(\Delta) + \|u_\epsilon^* - u^*\|_X$, (a) shows that

$$\sup_{f\in\Delta} \|f - u^*\|_X \leq \liminf_{\epsilon\to 0} r_{X,\mathbf{U}_\epsilon}(\Delta). \tag{5.2}$$

On the other hand, as $r_{X,\mathbf{U}_\epsilon}(\Delta) = \sup_{f\in\Delta} \|f - u_\epsilon^*\|_X \leq \sup_{f\in\Delta} \|f - u^*\|_X + \|u_\epsilon^* - u^*\|_X$, we have

$$\limsup_{\epsilon\to 0} r_{X,\mathbf{U}_\epsilon}(\Delta) \leq \sup_{f\in\Delta} \|f - u^*\|_X. \tag{5.3}$$

Therefore, (5.2) and (5.3) lead to

$$\lim_{\epsilon\to 0} r_{X,\mathbf{U}_\epsilon}(\Delta) = \sup_{f\in\Delta} \|f - u^*\|_X \tag{5.4}$$

Now, let $d = (d_1, \ldots, d_n) \in \mathbb{R}^n$ and $u^\epsilon = \sum_{j=1}^{n} d_j u_j^\epsilon$. It is clear that $\lim_{\epsilon\to 0} r_{X,\mathbf{U}_\epsilon}(\Delta) \leq \sup_{f\in\Delta} \left\| f - \sum_{j=1}^{n} d_j u_j^0 \right\|_X$. As d is arbitrary, it follows that $\lim_{\epsilon\to 0} r_{X,\mathbf{U}_\epsilon}(\Delta) \leq r_{X,\mathbf{U}_0}(\Delta)$. Finally (5.4) proves (b) and (c). $\square$

Below, we provide our main result of this section. Precisely, we extend an alternation theorem (Theorem 2.13) in the best approximation from weak Chebyshev spaces on a compact set of $\mathbb{R}$ to those in relative Chebyshev center on any compact set of $\mathbb{R}$. Furthermore, we give an intrinsic characterization of those linear spaces for which an alternation theorem holds. Additionally, this result is further considered to include the Chebyshev center relative to SH-spaces (Theorem 4.13).

Theorem 5.2. *Let $U \subset \mathcal{C}(M)$ be an n-dimensional linear space. The following statements are equivalent:*

(a) U is an n-dimensional WT-space on M;

(b) for each complete set Δ of uniformly bounded functions in $\mathcal{C}(M)$, either $\frac{1}{2}\|f_- - f_+\|_M = r_{M,U}(\Delta)$, or there exist at least one $u^* \in Z_{M,U}(\Delta)$, $\sigma \in \{-1, 1\}$, $n+1$ elements $z_1 < z_2 < \ldots < z_{n+1}$ in M, and $n+1$ functions $h_1, \ldots, h_{n+1} \in \{f_-, f_+\}$ such that

$$h_i(z_i) - u^*(z_i) = \sigma(-1)^{n+1-i} r_{M,U}(\Delta), \quad 1 \leq i \leq n+1.$$

Moreover, in the case $\frac{1}{2}\|f_- - f_+\|_M < r_{M,U}(\Delta)$,

$$\left(\frac{f_- + f_+}{2} - u^*\right)(z_i)\left(\frac{f_- + f_+}{2} - u^*\right)(z_{i+1}) < 0, \quad 1 \leq i \leq n. \tag{5.5}$$

Proof. Assume that U is an n-dimensional WT-space on M, and let Δ be a complete set of uniformly bounded functions in $\mathcal{C}(M)$. Suppose that $\frac{1}{2}\|f_- - f_+\|_M < r_{M,U}(\Delta)$ and let $0 < \epsilon < 1$. By hypothesis, there exists a basis $\{u_1, \ldots, u_n\}$ of U that is WT-system on M. From Theorem 3.5, we see that for every $0 < \epsilon < 1$, there exists a SH-system $\{u_1^\epsilon, \ldots, u_n^\epsilon\}$ on M such that

$$\lim_{\epsilon \to 0} \|u_j^\epsilon - u_j\|_M = 0, \quad 1 \leq j \leq n.$$

Let $U_\epsilon = \operatorname{span}\{u_1^\epsilon, \ldots, u_n^\epsilon\}$. By Corollary 4.12, it follows that $Z_{M,U_\epsilon}(\Delta)$ is a unitary set. We put $Z_{M,U_\epsilon}(\Delta) = \{u_\epsilon^*\}$. According to Theorem 5.1, there exists $u^* \in Z_{M,U}(\Delta)$ verifying

$$\lim_{\epsilon \to 0} \|u_\epsilon^* - u^*\|_M = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} r_{M,U_\epsilon}(\Delta) = r_{M,U}(\Delta). \tag{5.6}$$

Hence, there is $0 < \epsilon_0 < 1$ such that $\frac{1}{2}\|f_- - f_+\|_M < r_{M,U_\epsilon}(\Delta)$ for all $0 < \epsilon < \epsilon_0$. Let $0 < \epsilon < \epsilon_0$. Corollary 4.14 implies that there exist $\sigma_\epsilon \in \{-1, 1\}$, $n+1$ elements $z_1(\epsilon) < z_2(\epsilon) < \ldots < z_{n+1}(\epsilon)$ in M and $n+1$ functions $h_1^\epsilon, \ldots, h_{n+1}^\epsilon \in \{f_-, f_+\}$ such that

$$h_i^\epsilon(z_i(\epsilon)) - u_\epsilon^*(z_i(\epsilon)) = \sigma_\epsilon(-1)^{n+1-i} r_{M,U_\epsilon}(\Delta), \quad 1 \leq i \leq n+1, \tag{5.7}$$

and

$$\left(\frac{f_- + f_+}{2} - u_\epsilon^*\right)(z_i(\epsilon))\left(\frac{f_- + f_+}{2} - u_\epsilon^*\right)(z_{i+1}(\epsilon)) < 0, \tag{5.8}$$

$1 \leq i \leq n$. Since the net $\{(z_1(\epsilon), \ldots, z_n(\epsilon)) : 0 < \epsilon < \epsilon_0\}$ is contained in M^n, we can find a subnet, which we denote other time with the same index ϵ, such that

(a) $z_i(\epsilon)$ converges as $\epsilon \to 0$ to $z_i \in M$, $1 \leq i \leq n+1$,

(b) $z_i \leq z_{i+1}$, $1 \leq i \leq n$,

(c) $\sigma_\epsilon = \sigma \in \{-1, 1\}$, for all ϵ,

(d) $h_i^\epsilon = h_i \in \{f_-, f_+\}$, for all ϵ.

Therefore, according to (5.6)-(5.8) we obtain

$$h_i(z_i) - u^*(z_i) = \sigma(-1)^{n+1-i} r_{M,U}(\Delta), \quad 1 \leq i \leq n+1,$$

and

$$\left(\frac{f_- + f_+}{2} - u^*\right)(z_i)\left(\frac{f_- + f_+}{2} - u^*\right)(z_{i+1}) \leq 0, \quad 1 \leq i \leq n.$$

We claim that

$$u^*(z_i) \neq \left(\frac{f_- + f_+}{2}\right)(z_i), \quad 1 \leq i \leq n+1.$$

Indeed, on the contrary, we suppose that there is $1 \leq i \leq n+1$ such that $u^*(z_i) = \left(\frac{f_- + f_+}{2}\right)(z_i)$, then $\frac{1}{2}\|f_- - f_+\|_M < r_{M,U}(\Delta) = \frac{1}{2}|(f_- - f_+)(z_i)|$, which is a contradiction.

Consequently, we deduce that (5.5) holds and $z_i < z_{i+1}$.

Reciprocally, suppose (b) holds, and let $f \in \mathcal{C}(M)$. If $f \notin U$, then the complete set $\Delta = \{f\}$ satisfies $\frac{1}{2}\|f_- - f_+\|_M = 0 < r_{M,U}(\Delta) = r_{M,U}(f)$. By assumption there exist at least one $u^* \in Z_{M,U}(f)$, $\sigma \in \{-1, 1\}$, and $n+1$ elements $z_1 < z_2 < \ldots < z_{n+1}$ in M, such that

$$f(z_i) - u^*(z_i) = \sigma(-1)^{n+1-i} r_{M,U}(f), \quad 1 \leq i \leq n+1.$$

On the other hand, if $f \in U$, such a result remains valid. Therefore, Theorem 2.13 implies that U is an n-dimensional WT-space on M. This completes the proof. □

As an immediate consequence of Lemma 3.2 and Theorems 3.3 and 5.2, we obtain the following result about interpolation of relative Chebyshev centers.

Corollary 5.3. *Let Δ be a complete set of uniformly bounded functions in $\mathcal{C}(M)$ and let $U \subset \mathcal{C}(M)$ be an n-dimensional WT-space on M. Then, either $\frac{1}{2}\|f_- - f_+\|_M = r_{M.U}(\Delta)$, or there exist at least one $u^* \in Z_{M,U}(\Delta)$ and n elements $z_1 < z_2 < \ldots < z_n$ in M^* such that $T_M(u^*)$ interpolates to $T_M\left(\frac{f_- + f_+}{2}\right)$ at $z_1, \ldots, z_n$. In particular, u^* interpolates to $\frac{f_- + f_+}{2}$ at $z_1, \ldots, z_n$ in M provided that M is a compact interval of $\mathbb{R}$.*

For each $g \in \mathcal{C}(X)$, let

$$\mathcal{F}_X(g) := \{z \in X : |g(z)| = \|g\|_X\} \neq \emptyset.$$

When the relative Chebyshev radius and the Chebyshev radius coincide, we can derive a result similar to [3, Corollary 1.8]. Before we remember that, $z \in X$ is a straddle point of $(u^*; f_-, f_+)$-approximation if $f_+(z) - u^*(z) = u^*(z) - f_-(z) = r_{X,\mathbf{U}}(\Delta)$.

Theorem 5.4. *Let Δ be a complete set of uniformly bounded functions in $\mathcal{C}(X)$ and let $\mathbf{U} \subset \mathcal{C}(X)$ be an n-dimensional linear space. The following statements are equivalent:*

(a) $\frac{1}{2}\|f_- - f_+\|_X = r_{X,\mathbf{U}}(\Delta)$,

(b) for each $u^* \in Z_{X,\mathbf{U}}(\Delta)$ and $z \in \mathcal{F}_X(f_- - f_+)$, z is a straddle point of $(u^*; f_-, f_+)$-approximation.

In this case, we have $u^*(z) = \left(\frac{f_- + f_+}{2}\right)(z)$ for each $u^* \in Z_{X,\mathbf{U}}(\Delta)$ and $z \in \mathcal{F}_X(f_- - f_+)$. Further, if $z_1, z_2, \ldots, z_n$ in $\mathcal{F}_X(f_- - f_+)$ are distinct elements, then for each $u^* \in Z_{X,\mathbf{U}}(\Delta)$, u^* interpolates to $\frac{f_- + f_+}{2}$ at $z_1, \ldots, z_n$ in X.

Proof. $(b) \Longrightarrow (a)$ is obvious.

$(a) \Longrightarrow (b)$ Let $u^* \in Z_{X,\mathbf{U}}(\Delta)$ and let $z \in \mathcal{F}_X(f_- - f_+)$, that is, $|f_-(z) - f_+(z)| = \|f_- - f_+\|_X$. Let $h \in \{f_-, f_+\}$ be such that $|h(z) - u^*(z)| = \max\{|f_-(z) - u^*(z)|, |f_+(z) - u^*(z)|\}$. From (a) and (2.4), it follows that

$$\begin{aligned}\frac{1}{2}\|f_- - f_+\|_X = \frac{1}{2}|f_-(z) - f_+(z)| \leq \frac{1}{2}(|f_-(z) - u^*(z)| + |f_+(z) - u^*(z)|) \leq |h(z) - u^*(z)| \\ \leq \|h - u^*\|_X \leq r_{X,\mathbf{U}}(f_-, f_+) = r_{X,\mathbf{U}}(\Delta) = \frac{1}{2}\|f_- - f_+\|_X.\end{aligned}$$

Therefore, $\frac{1}{2}|f_-(z) - f_+(z)| = r_{X,\mathbf{U}}(\Delta)$. Further, as $\frac{1}{2}(|f_-(z) - u^*(z)| + |f_+(z) - u^*(z)|) = |h(z) - u^*(z)|$, we get $\frac{1}{2}|f_-(z) - f_+(z)| = |f_-(z) - u^*(z)| = |f_+(z) - u^*(z)|$, and so, $u^*(z) = \left(\frac{f_- + f_+}{2}\right)(z)$. This proves (b). The rest of the proof follows easily. □

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