Best L^2 Local Approximation On Two Small Intervals. *

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Abstract

In this paper we extends the concept of L^2 differentiable functions. Consider functions differentiable in this new sense and we prove the existence of the best local approximation polynomial on two points in $\mathbb{R}$, from the space of polynomials of degree n. Moreover, we give a characterization of the best local approximation polynomial.

1 Introduction.

Let $x_1 \in \mathbb{R}, x_1 \neq 0, x_2 = -x_1$, and let $a > 0$ be such that the intervals $I_{a,i} := [x_i - a, x_i + a]$, $1 \leq i \leq 2$, are pairwise disjoint. Let $\mathcal{L}$ be the space of equivalence class of Lebesgue measurable real functions defined on $I_a := I_{a,1} \cup I_{a,2}$. For each Lebesgue measurable set $A \subset I_a$, with $|A| > 0$, we consider the semi-norms on $\mathcal{L}$,

$$\|f\|_A := \left(|A|^{-1} \int_A |f(x)|^2 dx \right)^{1/2},$$

where $|A|$ denotes the measure of the set A.
If $0 < \epsilon \leq a$, we denote $I_{\epsilon,i} = [x_i - \epsilon, x_i + \epsilon]$, $\|f\|_{\epsilon,i} = \|f\|_{I_{\epsilon,i}}$ and $\|f\|_\epsilon = \|f\|_{I_\epsilon}$
For a non negative integer s, let Π^s be the linear space of polynomials of degree at most s.

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Henceforward, we consider $n, q, r \in \mathbb{N} \cup \{0\}$ such that $n+1 = 2q+r, r < 2$. If $f \in L^2(I_\epsilon)$, it is well known that there exists a unique best $\|.\|_\epsilon$ approximation of f from Π^n, say $P_\epsilon(f)$, satisfying

$$\|f - P_\epsilon(f)\|_\epsilon \leq \|f - P\|_\epsilon, \quad \text{for all} \quad P \in \Pi^n.$$

If f is an even (odd) function, then $P_\epsilon(f)$ is an even (odd) polynomial. $P_\epsilon(f)$ is characterized by

$$\int_{I_\epsilon} (f - P_\epsilon(f))(x)P(x)dx = 0, \quad \text{for all} \quad P \in \Pi^n. \tag{1.1}$$

If $\lim_{\epsilon \to 0} P_\epsilon(f)$ exists, say $P_0(f)$, it is called the *best local approximation of f on $\{x_1, x_2\}$ from Π^n.*

We recall that a function $f \in L^2(I_{a,i})$ is L^2 *differentiable of order s at x_i* if there exists $Q_i \in \Pi^s$ such that

$$\|f - Q_i\|_{\epsilon,i} = o(\epsilon^s) \tag{1.2}$$

The class of these functions is denoted by $t^2_s(x_i)$. We also write $t^2_{-1}(x_i) = L^2(I_{a,i})$. This concept was introduced by Calderón and Zygmund in [1]. It is well known that there exists at most one polynomial verifying 1.2 (see [10]).

The problem of best local approximation was formally introduced and studied in a paper by Chui, Shisha and Smith [2]. However, the initiation of this could be dated back to results of J.L. Walsh [12], who proved that the Taylor polynomial of an analytic function f over a domain is the limit of the net of polynomial best approximations of a given degree, by shrinking the domain to a single point. In [11] this problem was considered for certain class of functions differentiable in ordinary sense on two points. Later, in [10] the author extended it for L^2 differentiable functions. In recent papers [5]-[9] was considered the problem of existence of the best local approximation for different norms, for certain class of differentiable functions and more general domains.

All the exponents in this work will be non negative integers number. We introduce the following definition.

Definition 1.1. *A function $f \in L^2(I_{a,i})$ satisfies the τ condition of order s at x_i, if there exists $Q_i \in \Pi^s$ such that*

$$\int_{x_i-\epsilon}^{x_i+\epsilon} (f - Q_i)(x)(x^2 - x_i^2)^j dx = o(\epsilon^{s+j+1}),\ 0 \leq j \leq s, \epsilon \to 0. \tag{1.3}$$

If f verifies (1.3), we say that $f \in \tau_s(x_i)$.

Let $\tau_s(\pm x_i) := \tau_s(x_i) \cap \tau_s(-x_i)$ and $t_s^2(\pm x_i) := t_s^2(x_i) \cap t_s^2(-x_i)$. The next theorem shows the uniqueness in Definition 1.1.

Theorem 1.2. *Let $f \in L^2(I_{a,i})$. Then there exists at most a polynomial $Q_i \in \Pi^s$ satisfying (1.3).*

Proof. Suppose that $Q_i, \overline{Q}_i \in \Pi^s$ verify (1.3). It is easy to see that $T(x) = (Q_i - \overline{Q}_i)(x) := \sum_{m=0}^{s} a_m (x - x_i)^m$ satisfies

$$\sum_{m=0}^{s} a_m \int_{I_{\epsilon,i}} (x - x_i)^{m+j} (x + x_i)^j dx = o(\epsilon^{s+j+1}),\ 0 \le j \le s,\ \epsilon \to 0. \tag{1.4}$$

We put $a_{-1} = 0$. If $a_m = 0$ for all m, $-1 \le m \le l < s$, then $a_{l+1} = 0$. In fact, considering $j = l + 1$ in (1.4), we get

$$\begin{aligned} a_{l+1} \int_{I_{\epsilon,i}} (x - x_i)^{2l+2} (x + x_i)^{l+1} dx \\ + \sum_{m=l+2}^{s} a_m \int_{I_{\epsilon,i}} (x - x_i)^{m+l+1} (x + x_i)^{l+1} dx = o(\epsilon^{s+l+2}). \end{aligned} \tag{1.5}$$

Since the summation in (1.5) is $O(\epsilon^{2l+4})$, we have

$$a_{l+1} \int_{I_{\epsilon,i}} (x - x_i)^{2l+2} (x + x_i)^{l+1} dx = o(\epsilon^{2l+3}). \tag{1.6}$$

Therefore,

$$a_{l+1} \int_{I_{\epsilon,i}} (x - x_i)^{2l+2} dx = o(\epsilon^{2l+3}), \text{ i.e., } a_{l+1} = 0.$$

This proves the lemma. □

By the previous theorem, if $f \in \tau_s(x_i)(\tau_s(-x_i))$, we denote $Q_{x_i}^s(f)(Q_{-x_i}^s(f))$ the unique polynomial of degree s verifying (1.3) at the points $x_i(-x_i)$. The proof of the next theorem immediately follows.

Theorem 1.3. *Let s be a non negative integer number. Then the operador $D_s : \tau_s(x_i) \to \Pi^s$ defined by $D_s(f)(x) = Q_{x_i}^s(f)(x)$, is linear. Moreover, $\tau_{s+1}(x_i) \subset \tau_s(x_i)$ and for $f \in \tau_{s+1}(x_i)$, $D_{s+1}(f)(x) = D_s(f)(x) + \alpha(x - x_i)^{s+1}$, $\alpha \in \mathbb{R}$.*

Now, if $f \in \tau_s(x_i)$ we can define the j-th τ derivative of f at x_i by

$$f^{(j)}(x_i) = (Q^s_{x_i}(f))^{(j)}(x_i),\ 0 \le j \le s. \tag{1.7}$$

Remark 1.4. We have $t^2_s(x_i) \subset \tau_s(x_i)$. In fact, using Hölder inequality we can see that the polynomial in Π^s that verifies $\|f - Q_i\|_{\epsilon,i} = o(\epsilon^s)$ also satisfies (1.3). In addition, the inclusion is strict as shows the following example. Let $f(x) = \sin(\frac{1}{x-x_i}), x \neq x_i$. It is easy to see that $f \in \tau_0(x_i)$. On other hand, if for some constant $\alpha \in \mathbb{R}$, $\|f - \alpha\|_{\epsilon,1} = o(1)$, then $\|f + \alpha\|_{\epsilon,i} = o(1)$, thus $\alpha = 0$. Now, a straightforward computation shows that $\|f\|_{\epsilon,i} \neq o(1)$, so $f \notin t^2_0(x_i)$.

In Section 2, we estimate the order of certain determinants depending on ϵ, and we prove some lemmas concerning to algebraic polynomials.

The main results of this paper are in Section 3. We prove that if either n is even, $f \in \tau_q(\pm x_i)$ and the odd part of f belongs to $t^2_{q-1}(x_i)$, or n is odd, $f \in \tau_{q-1}(\pm x_i)$ and the odd part of f belongs to $t^2_{q-2}(x_i)$, then there exists the best local approximation. Moreover, we give a characterization of it. Our theorems extend the mentioned results proved in [10] to a more wide class of functions.

2 Auxiliary results

We begin this section estimating the order of the determinant of certain matrix depending on ϵ.

Lemma 2.1. *Let $u \in \mathbb{N} \cup \{0\}$ and let $A = (a_{jl})$ be a matrix of order $(u+1) \times (u+1)$, with $a_{jl} = a_{jl}(\epsilon) := \int_{1-\epsilon}^{1+\epsilon} (x^2-1)^{j+l} w(x)dx$, $0 \le j, l \le u$, where w is a continuous function in a neighborhood of 1 such that $w(1) = 1$. Then the determinant of A, say $D(\epsilon)$, satisfies*

$$D(\epsilon) = (M + o(1))\epsilon^{(u+1)^2}, \tag{2.1}$$

where M is a non null constant.

Proof. A is the Gramian matrix of the set of independent linearly polynomials $\{(x^2-1)^j\}_{j=0}^u$ with the inner product $\langle \cdot, \cdot \rangle_{w,\epsilon}$ on $[1-\epsilon, 1+\epsilon]$, then $D(\epsilon) \neq 0$. For each pair j, l, the functions $(x \pm 1)^{j+l} w(x)$ is continuous and $(x-1)^{j+l}$ has a sign constant in $[1-\epsilon, 1)$ and $(1, 1+\epsilon]$, therefore by an applying of a

Value Mean Theorem for integrals there exist $\eta := \eta(\epsilon, j, l) \in [1, 1+\epsilon]$ and $\eta' := \eta'(\epsilon, j, l) \in [1-\epsilon, 1]$ such that

$$a_{jl} = w(\eta)(\eta+1)^{j+l} b_{jl} + w(\eta')(\eta'+1)^{j+l} b'_{jl},\ 0 \le j, l \le u, \tag{2.2}$$

where $b_{jl} = b_{jl}(\epsilon) := \int_1^{1+\epsilon} (x-1)^{j+l} dx$ y $b'_{jl} = b'_{jl}(\epsilon) := \int_{1-\epsilon}^{1} (x-1)^{j+l} dx$.
We observe that $a_{jl} = [2^{j+l} + o_{jl}(1)] b_{jl} + [2^{j+l} + o'_{jl}(1)] b'_{jl}$, where $o_{jl}(1), o'_{jl}(1)$ are functions of the variable ϵ which tend to zero as $\epsilon \to 0$.
It is well known that if p is an arbitrary permutation of the set $S = \{0, 1, \ldots, u\}$, then

$$\det(A) = \sum_p \mathrm{sg}(p) \prod_{j=0}^{u} a_{jp(j)}. \tag{2.3}$$

We consider the matrix $B = (b_{jl})$ and $B' = (b'_{jl})$. By Lemma 2.1 in [3],

$$\det(B) = \sum_p \mathrm{sg}(p) \prod_{j=0}^{u} b_{jp(j)} = \sum_p \mathrm{sg}(p) \prod_{j=0}^{u} b'_{jp(j)} = \det(B') = C\epsilon^{(u+1)^2}, \tag{2.4}$$

where C is a constant non null. On the other hand, expanding $\prod_{j=0}^{u} a_{jp(j)}$ in groups of terms containing only the factors b, only the factors b', and the mixed products, we have

$$\begin{aligned}\prod_{j=0}^{u} a_{jp(j)} &= \prod_{j=0}^{u} [2^{j+p(j)} + o_{jp(j)}(1)] b_{jp(j)} + \prod_{j=0}^{u} [2^{j+p(j)} + o'_{jp(j)}(1)] b'_{jp(j)} + K\epsilon^{(u+1)^2} \\ &= 2^{u(u+1)} \prod_{j=0}^{u} b_{jp(j)} + 2^{u(u+1)} \prod_{j=0}^{u} b'_{jp(j)} + K\epsilon^{(u+1)^2} + o_p(\epsilon^{(u+1)^2}),\end{aligned} \tag{2.5}$$

for some constant K. Then, from (2.3)-(2.5) we get $D(\epsilon) = (M + o(1))\epsilon^{(u+1)^2}$, where $M = 2^{u(u+1)+1} C + K$. □

Lemma 2.2. *Let $s, u \in \mathbb{N} \cup \{0\}, s \le u$. Let $C = (c_{jl})$ be the matrix of order $(u+1) \times (u+1)$ defined by*

$$c_{jl} := c_{jl}(\epsilon) = \begin{cases} \langle (x^2-1)^j, (x^2-1)^l \rangle_{w,\epsilon} & 0 \le j, l \le u,\ l \ne s, \\ \epsilon^{j+u+1} O_j(1) & 0 \le j \le u,\ l = s, \end{cases} \tag{2.6}$$

where w is as in Lemma 2.1 and $O_j(1)$ is a function of the variable ϵ which is bounded for $\epsilon \to 0$. Then the determinant of C, say $N(s,\epsilon)$, satisfies $N(s,\epsilon) = O(\epsilon^{u-s+(u+1)^2})$.
If we substitute $O_j(1)$ by $o_j(1)$, then $N(s,\epsilon) = o(\epsilon^{u-s+(u+1)^2}), 0 \le s \le u$.

Proof. Let C'_{jl} denote the sub matrix of C, where have omitted the j-th file and the l-th column. If we expand the determinant of C'_{jl} by the column $l = s$, we have

$$N(s,\epsilon) = \sum_{j=0}^{u} (-1)^{j+s} c_{js} \det(C'_{js}) = \sum_{j=0}^{u} \epsilon^{j+u+1} O_j(1) \det(C'_{js}). \tag{2.7}$$

Let $p := p_{js}$ be an arbitrary bijection of the set $\{0,\ldots,j-1,j+1,\ldots,u\}$ onto $\{0,\ldots,s-1,s+1,\ldots,u\}$. Then

$$\det(C'_{js}) = \sum_{p} \mathrm{sg}(p) \prod_{k=0,k\neq j}^{u} a_{kp(k)}, \tag{2.8}$$

where the elements $a_{kp(k)}$ were given in Lemma 2.1.
Multiplying by ϵ^{j+s+1} and its inverse in (2.8), from (2.7) we get

$$N(s,\epsilon) = \sum_{j=0}^{u} \epsilon^{u-s} O_j(1) \left[\epsilon^{j+s+1} \sum_{p} \mathrm{sg}(p) \prod_{k=0,k\neq j}^{u} a_{kp(k)} \right]. \tag{2.9}$$

Since the expression in the bracket is an algebraic sum of terms of the form $K\epsilon^{(u+1)^2}$, with K a constant, from (2.9) we obtain

$$N(s,\epsilon) = \sum_{j=0}^{u} \epsilon^{u-s} O'_j(1) \epsilon^{(u+1)^2} = O(\epsilon^{u-s}) \epsilon^{(u+1)^2}. \tag{2.10}$$

Finally, the last part of the lemma analogously follows to the above proof. □

As a consequence of the previous lemmas we get some results on nets of even polynomials.

Lemma 2.3. *Let $T_\epsilon(x) = \sum_{l=0}^{q-1} b_l(\epsilon) x^2 (x^2-1)^l$ be a net of polynomials in Π^{2q}, such that*

$$\int_{1-\epsilon}^{1+\epsilon} T_\epsilon(x)(x^2-1)^j dx = o(\epsilon^{q+j}),\ 0 \le j \le q-1, \tag{2.11}$$

then

$$b_l(\epsilon) = o(\epsilon^{q-l-1}),\ 0 \le l \le q-1. \tag{2.12}$$

In particular, the net $\{T_\epsilon\}_{\epsilon>0}$ *converges to zero as* $\epsilon \to 0$.

Proof. From (2.11) we have the following linear system,

$$\sum_{l=0}^{q-1} a_{jl}(\epsilon) b_l(\epsilon) = o(\epsilon^{q+j}),\ 0 \le j \le q-1, \tag{2.13}$$

where $a_{jl}(\epsilon)$ was introduced in Lemma 2.1 with $w(x) = x^2$. Now, applying Lemma 2.1 and Lemma 2.2 with $u = q-1$, and later the Cramer rule we obtain (2.12). □

Lemma 2.4. *Let* $T_\epsilon(x) = \sum_{l=0}^{q} b_l(\epsilon)(x^2-1)^l$ *be a net of polynomials in* Π^{2q}. *If*

$$\int_{1-\epsilon}^{1+\epsilon} T_\epsilon(x)(x^2-1)^j dx = O(\epsilon^{q+j+1}),\ 0 \le j \le q, \tag{2.14}$$

then for each $0 \le l \le q$,

$$b_l(\epsilon) = O(\epsilon^{q-l}) \quad \text{and} \quad T_\epsilon^{(l)}(\pm 1) = O(\epsilon^{q-l}). \tag{2.15}$$

In particular, the net $\{T_\epsilon\}_{\epsilon>0}$ *is uniformly bounded on compact sets as* $\epsilon \to 0$.

Proof. From (2.14) we have the following linear system,

$$\sum_{l=0}^{q} a_{jl}(\epsilon) b_l(\epsilon) = O(\epsilon^{q+j+1}),\ 0 \le j \le q, \tag{2.16}$$

where $a_{jl}(\epsilon)$ was introduced in Lemma 2.1 with $w = 1$. Now, applying Lemma 2.1 and Lemma 2.2 with $u = q$, and later the Cramer rule we obtain $b_l(\epsilon) = O(\epsilon^{q-l}), 0 \le l \le q$. On the other hand, Leibnitz rule implies that

$$T_\epsilon^{(s)}(1) = \sum_{l=0}^{q} b_l(\epsilon) \sum_{m=0}^{s} \binom{s}{m} [(x-1)^l]^{(m)}(1)[(x+1)^l]^{(s-m)}(1)$$

$$= \sum_{l=0}^{s} b_l(\epsilon) \binom{s}{l} l! [(x+1)^l]^{(s-l)}(1) = O(\epsilon^{q-s}), 0 \le s \le q,$$

where the last equality is a consequence of Theorem 2.4.
Since T_ϵ is even, then $T_\epsilon^{(s)}(-1) = (-1)^s T_\epsilon^{(s)}(1) = O(\epsilon^{q-s}), 0 \le s \le q$. □

An analogously proof to previous lemma with $u = q-1$ yields the next lemma.

Lemma 2.5. *Let* $T_\epsilon(x) = \sum_{l=0}^{q-1} b_l(\epsilon)(x^2-1)^l$ *be a net of polynomials in* Π^{2q-2}. *If*

$$\int_{1-\epsilon}^{1+\epsilon} T_\epsilon(x)(x^2-1)^j dx = o(\epsilon^{q+j}),\ 0 \le j \le q-1, \tag{2.17}$$

then $b_l(\epsilon) = o(\epsilon^{q-l-1})$, $0 \le l \le q-1$. *In particular, the net* $\{T_\epsilon\}_{\epsilon>0}$ *converges to* 0, *as* $\epsilon \to 0$.

3 Existence of the best local approximation.

We prove the existence of best local approximation. Without loss of generality, we only must consider $x_1 = 1$. In fact, it is sufficient to do the variable change $t = x/x_i$ in (1.3).

3.1 The case n even

In this subsection we assume n even, i.e., $r = 1$, and $f \in \tau_q(\pm 1)$. For $q \ge 1$, we define the following set

$$\mathcal{S}(f) = \{H \in \Pi^{2q} : H^{(j)}(\pm 1) = f^{(j)}(\pm 1),\ 0 \le j \le q-1\}.$$

Let $S_0 \in \mathcal{S}(f)$ be a fixed polynomial. Then any polynomial in $\mathcal{S}(f)$ can be written as $S_0(x) + \lambda(x^2-1)^q, \lambda \in \mathbb{R}$. If $q = 0$ we put $\mathcal{S}(f) = \Pi^0$.

We consider the function $g = f - S_0$. According to (1.7) and Theorem 1.3, it is easy to see that

$$\mathcal{S}(f) = S_0 + \mathcal{S}(g), \quad g \in \tau_q(\pm 1), \quad g^{(j)}(\pm 1) = 0, 0 \le j \le q-1. \tag{3.1}$$

The proofs of the following lemmas are immediate.

Lemma 3.1. *It verifies that* $P_\epsilon(f) = S_0 + P_\epsilon(g)$. *In addition, if* $P_0(g)$ *exists then* $P_0(f)$ *exists and* $P_0(f) = S_0 + P_0(g)$.

Lemma 3.2. *Let* $P \in \Pi^{2q}$ *be an even polynomial. Then there exist unique even polynomials* $U \in \mathcal{S}(g)$ *and* $S \in \Pi^{2q-2}$ *such that* $P = U + S$.

Now, our purpose is to prove the existence and characterization of $P_0(g)$. We consider the even and odd parts of g, i.e., $g^e(x) = \frac{g(x)+g(-x)}{2}$ and $g^o(x) = \frac{g(x)-g(-x)}{2}$, respectively. If there exist $P_0(g^e)$ and $P_0(g^o)$, clearly $P_0(g) = P_0(g^e) + P_0(g^o)$.

Lemma 3.3. *It verifies that $g^e, g^o \in \tau_q(\pm 1)$.*

Proof. By (3.1), $g \in \tau_q(\pm 1)$, and $Q^q_{\pm 1}(g)(x) = \alpha_{\pm 1}(x \mp 1)^q$ for some real numbers α_{-1}, α_{+1}, which verify

a) $\int_{1-\epsilon}^{1+\epsilon}(g(x) - \alpha_{+1}(x-1)^q)(x^2-1)^j dx = o(\epsilon^{q+j+1})$, $0 \le j \le q$.

b) $\int_{-1-\epsilon}^{-1+\epsilon}(g(x) - \alpha_{-1}(x+1)^q)(x^2-1)^j dx = o(\epsilon^{q+j+1})$, $0 \le j \le q$.

If in b) we do the change of variable x by $-x$, and then add the equation a) member to member, we obtain

$$\int_{1-\epsilon}^{1+\epsilon}(g^e(x) - \gamma_{+1}(x-1)^q)(x^2-1)^j dx = o(\epsilon^{q+j+1}),\ 0 \le j \le q, \tag{3.2}$$

where $\gamma_{+1} := \frac{\alpha_{+1}+(-1)^q\alpha_{-1}}{2}$. So, $Q^q_{+1}(g^e)(x) = \gamma_{+1}(x-1)^q$ and $g^e \in \tau_q(+1)$. An analogous proof with the polynomials

$$Q^q_{-1}(g^e)(x) = \gamma_{-1}(x+1)^q, \quad Q^q_{\pm 1}(g^o)(x) = \beta_{\pm 1}(x \mp 1)^q, \tag{3.3}$$

where $\gamma_{-1} := \frac{(-1)^q\alpha_{+1}+\alpha_{-1}}{2}, \beta_{-1} := \frac{(-1)^q\alpha_{+1}-\alpha_{-1}}{2}, \beta_{+1} := \frac{\alpha_{+1}-(-1)^q\alpha_{-1}}{2}$, yields $g^e \in \tau_q(-1)$ and $g^o \in \tau_q(\pm 1)$. □

Proposition 3.4. *If $g^o \in t^2_{q-1}(1)$, then $P_0(g^o) = 0$.*

Proof. According to (3.3) we have

$$\int_{1-\epsilon}^{1+\epsilon}(g^o(x) - \beta_{+1}(x-1)^q)(x^2-1)^j dx = o(\epsilon^{q+j+1}),\ 0 \le j \le q. \tag{3.4}$$

Therefore

$$\int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j dx = o(\epsilon^{q+j}),\ 0 \le j \le q-1, \tag{3.5}$$

In consequence $Q^{q-1}_1(g^o) = 0$ and $g^o \in \tau_{q-1}(1)$. Let $Q \in \Pi^{q-1}$ be such that $\|g^o - Q\|_{\epsilon,1} = o(\epsilon^{q-1})$. By Remark 1.4 and Theorem 1.2, $Q = 0$.
By Hölder inequality we obtain

$$\left|\int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j x dx\right| \le K\|g^o\|_{\epsilon,1}\epsilon^{j+1} = o(\epsilon^{q+j}),\ 0 \le j \le q-1. \tag{3.6}$$

On the other hand, from the characterization of $P_\epsilon(g^o)$

$$\int_{1-\epsilon}^{1+\epsilon} (g^o - P_\epsilon(g^o))(x)(x^2-1)^j x dx = 0,\ 0 \le j \le q-1. \tag{3.7}$$

In fact, it is a consequence that the integrand is an even function. From (3.6) and (3.7) follows that

$$\int_{1-\epsilon}^{1+\epsilon} x P_\epsilon(g^o)(x)(x^2-1)^j dx = o(\epsilon^{q+j}), 0 \le j \le q-1. \tag{3.8}$$

Since $P_\epsilon(g^o) \in \Pi^{2q-1}$ is a odd polynomial and $\{(x^2-1)^j x\}_{j=0}^{q-1}$ is a basis, we can write

$$P_\epsilon(g^o)(x) = \sum_{l=0}^{q-1} b_l(\epsilon)(x^2-1)^l x.$$

Therefore, (3.8) and Lemma 2.3 imply that $P_\epsilon(g^o) \to 0$, as $\epsilon \to 0$. □

Proposition 3.5. *The net of polynomials $\{P_\epsilon(g^e)\}_{\epsilon>0}$, for small ϵ, is uniformly bounded on compact sets.*

Proof. From (3.2) we get

$$\int_{1-\epsilon}^{1+\epsilon} g^e(x)(x^2-1)^j dx = O(\epsilon^{q+j+1}),\ 0 \le j \le q. \tag{3.9}$$

Now, (1.1) implies that

$$\int_{1-\epsilon}^{1+\epsilon} (g^e - P_\epsilon(g^e))(x)(x^2-1)^j dx = 0,\ 0 \le j \le q, \tag{3.10}$$

because the integrand is an even function. From (3.9) and (3.10) follows that

$$\int_{1-\epsilon}^{1+\epsilon} P_\epsilon(g^e)(x)(x^2-1)^j dx = O(\epsilon^{q+j+1}),\ 0 \le j \le q. \tag{3.11}$$

Expanding $P_\epsilon(g^e)$ in terms of the basis $\{(x^2-1)^j\}_{j=0}^{q}$, from (3.11) and Lemma 2.4 follows that $\{P_\epsilon(g^e)\}_{\epsilon>0}$, for small ϵ, is uniformly bounded on compact sets. □

Given a polynomial $P \in \Pi^{2q}$, let $P^* \in \Pi^q$ be defined by

$$P^*(x) = \gamma_{+1}(x-1)^q - q!^{-1}U^{(q)}(1)(x-1)^q - \sum_{l=0}^{q-1} l!^{-1}P^{(l)}(1)(x-1)^l, \quad (3.12)$$

where U is the polynomial mentioned in Lemma 3.2 and γ_{+1} was introduced in (3.2). If $q = 0$, in (3.12) we omit the last term.

We consider the linear functional $F : L^2([0,1]) \times \Pi^{2q} \to \mathbb{R}$ defined by

$$F(h, P) = \int_{1-\epsilon}^{1+\epsilon} (h - P)(x) \frac{(x^2-1)^q}{\epsilon^{2q+1}} dx. \quad (3.13)$$

Lemma 3.6. *Let* $\{P_\epsilon(g^e) = U_\epsilon + S_\epsilon\}_{\epsilon>0} \subset \Pi^{2q}$ *be a net of polynomials where* U_ϵ *and* S_ϵ *are as in Lemma 3.2. Then* $S_\epsilon \to 0$ *and* $F(0, P_\epsilon(g^e)^*) = o(1)$.

Proof. Clearly $P_\epsilon(g^e)^{(j)}(\pm 1) = S_\epsilon^{(j)}(\pm 1), 0 \le j \le q-1$. $P_\epsilon(g^e)$ is an even polynomial satisfying (3.11), thus Lemma 2.4 implies that $S_\epsilon^{(j)}(\pm 1) = O(\epsilon^{q-j})$, $0 \le j \le q-1$. Since $S_\epsilon \in \Pi^{2q-2}$, $S_\epsilon \to 0$.
On the other hand,

$$\begin{aligned} F(g^e, P_\epsilon(g^e) + P_\epsilon(g^e)^*) &= \int_{1-\epsilon}^{1+\epsilon} (g^e(x) - \gamma_{+1}(x-1)^q) \frac{(x^2-1)^q}{\epsilon^{2q+1}} dx \\ &\quad - \int_{1-\epsilon}^{1+\epsilon} q!^{-1} S_\epsilon^{(q)}(1)(x-1)^q \frac{(x^2-1)^q}{\epsilon^{2q+1}} dx \\ - \int_{1-\epsilon}^{1+\epsilon} \sum_{l=q+1}^{2q} & l!^{-1} P_\epsilon(g^e)^{(l)}(1)(x-1)^l \frac{(x^2-1)^q}{\epsilon^{2q+1}} dx. \end{aligned} \quad (3.14)$$

By (3.2) and mean of the variable change $x = 1 + \epsilon t$ in (3.14), we get

$$\begin{aligned} F(g^e, P_\epsilon(g^e) + P_\epsilon(g^e)^*) &= o(1) - \int_{-1}^{1} q!^{-1} S_\epsilon^{(q)}(1) t^{2q}(2+\epsilon t)^q dt \\ &\quad - \sum_{l=q+1}^{2q} \epsilon^{l-q} l!^{-1} P_\epsilon(g^e)^{(l)}(1) \int_{-1}^{1} t^{2q}(2+\epsilon t)^q dt. \end{aligned} \quad (3.15)$$

Proposition 3.5 implies that $P_\epsilon(g^e) = O(1)$ as $\epsilon \to 0$. Therefore, since $S_\epsilon \to 0$ we obtain $F(g^e, P_\epsilon(g^e) + P_\epsilon(g^e)^*) = o(1)$. In addition, from (3.10) follows that $F(g^e, P_\epsilon(g^e)) = 0$. In consequence, we get $F(0, P_\epsilon(g^e)^*) = o(1)$. □

Now, we establish one of our main results.

Theorem 3.7. *Let $n = 2q$, $f \in \tau_q(\pm 1)$ and $f^o \in t^2_{q-1}(1)$. Then there exists the best local approximation of f on $\{-1, 1\}$ from Π^n. Moreover, if $S_0 \in \mathcal{S}(f)$ and $g = f - S_0$ then*

$$P_0(f)(x) = S_0(x) + \frac{g^{(q)}(1) + (-1)^q g^{(q)}(-1)}{q!2^{q+1}}(x^2 - 1)^q. \tag{3.16}$$

Proof. Since $f^o \in t^2_{q-1}(1)$ then $g^o \in t^2_{q-1}(1)$. So, Proposition 3.4 implies that $P_0(g^o) = 0$. Therefore, it is sufficient to find $P_0(g^e)$. From Lemma 3.6 and Lemma 2.4 we have

$$F(0, P_\epsilon(g^e)^*) = o(1), \quad \text{and} \quad \epsilon^{l-q} P_\epsilon(g^e)^{(l)}(1) = O(1),\ 0 \le l \le q, \tag{3.17}$$

respectively. Since $U_\epsilon \in \mathcal{S}(g^e)$, we can write $U_\epsilon(x) = \lambda_\epsilon(x^2 - 1)^q, \lambda_\epsilon \in \mathbb{R}$. In consequence, from (3.13) and (3.17) we conclude

$$\int_{-1}^{1} (2^q \lambda_\epsilon - \gamma_{+1}) t^{2q} (2 + \epsilon t)^q dt = o(1). \tag{3.18}$$

Proposition 3.5 implies that $P_\epsilon(g^e) = O(1)$. Further, by Lemma 3.6 $S_\epsilon \to 0$, thus $\lambda_\epsilon = O(1)$. Now, if $\{\lambda_{\epsilon_m}\}$ is a sequence converging to λ_0, from (3.18) we get

$$\int_{-1}^{1} (2^q \lambda_0 - \gamma_{+1}) t^{2q} 2^q dt = 0, \tag{3.19}$$

i.e.,

$$\lambda_0 = \frac{\gamma_{+1}}{2^q} = \frac{g^{(q)}(1) + (-1)^q g^{(q)}(-1)}{q!2^{q+1}}. \tag{3.20}$$

Therefore, the net $\{\lambda_\epsilon\}_{\epsilon>0}$ converges to λ_0, i.e., $P_\epsilon(g^e) \to \frac{\gamma_{+1}}{2^q}(x^2 - 1)^q$. Finally, (3.1) implies (3.16). □

3.2 The case n odd

In this subsection we assume n odd, i.e., $r = 0$, and $f \in \tau_{q-1}(\pm 1)$. Let $R_0 \in \Pi^{2q-1}$ be the polynomial determined by the conditions $R_0^{(j)}(\pm 1) = f^{(j)}(\pm 1)$, $0 \le j \le q - 1$, and let $g = f - R_0$. According to (1.7) and Theorem 1.3, it is easy to see that

$$g \in \tau_{q-1}(\pm 1), \quad g^{(j)}(\pm 1) = 0, 0 \le j \le q - 1. \tag{3.21}$$

Remark 3.8. We observe that Lemma 3.1 holds with R_0 instead of S_0. Further by (3.21), $Q_{\pm1}^{q-1}(g) = 0$.

Using Remark 3.8, with an analogously proof to Lemma 3.3 we get the next lemma.

Lemma 3.9. *It verifies that* $g^e, g^o \in \tau_{q-1}(\pm1)$ *with* $Q_{\pm1}^{q-1}(g^e) = Q_{\pm1}^{q-1}(g^o) = 0$.

Proposition 3.10. *If* $g^o \in t_{q-2}^2(1)$, *then* $P_0(g^e) = P_0(g^o) = 0$.

Proof. From Lemma 3.9 we get

$$\int_{1-\epsilon}^{1+\epsilon} g^e(x)(x^2-1)^j dx = o(\epsilon^{q+j}),\ 0 \le j \le q-1. \tag{3.22}$$

Now, (1.1) implies that

$$\int_{1-\epsilon}^{1+\epsilon} (g^e - P_\epsilon(g^e))(x)(x^2-1)^j dx = 0,\ 0 \le j \le q-1, \tag{3.23}$$

because the integrand is an even function. From (3.22) and (3.23) follows that

$$\int_{1-\epsilon}^{1+\epsilon} P_\epsilon(g^e)(x)(x^2-1)^j dx = o(\epsilon^{q+j}),\ 0 \le j \le q-1. \tag{3.24}$$

Since n is odd, then $P_\epsilon(g^e)$ is an even polynomial in Π^{2q-2}. Expanding $P_\epsilon(g^e)$ in terms of the basis $\{(x^2-1)^j\}_{j=0}^{q-1}$, from (3.24) and Lemma 2.5 it follows that $P_0(g^e) = 0$.

Next, we prove that $P_0(g^o) = 0$. Since $g^o \in t_{q-2}^2(1)$, there exists $Q \in \Pi^{q-2}$ such that $\|g^o - Q\|_{\epsilon,1} = o(\epsilon^{q-2})$. By Remark 1.4, $t_{q-2}^2(1) \subset \tau_{q-2}(1)$. From Theorem 1.3 and Lemma 3.9 we get $Q = Q_1^{q-2}(g^o) = 0$.
By Hölder inequality we obtain

$$\left| \int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j (x-1) dx \right| \le K \|g^o\|_{\epsilon,1} \epsilon^{j+2} = o(\epsilon^{q+j}),\ 0 \le j \le q-1, \tag{3.25}$$

for some constant K. By Lemma 3.9,

$$\int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j dx = o(\epsilon^{q+j}),\ 0 \le j \le q-1. \tag{3.26}$$

From (3.25) and (3.26) we have,

$$\begin{aligned}\int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j x dx = \int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j (x-1) dx \\ + \int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j dx = o(\epsilon^{q+j}).\end{aligned} \tag{3.27}$$

On the other hand, (1.1) implies that

$$\int_{1-\epsilon}^{1+\epsilon} (g^o - P_\epsilon(g^o))(x)(x^2-1)^j x dx = 0,\ 0 \le j \le q-1, \tag{3.28}$$

because the integrand is an even function. From (3.27) and (3.28) we get

$$\int_{1-\epsilon}^{1+\epsilon} P_\epsilon(g^o)(x)(x^2-1)^j x dx = o(\epsilon^{q+j}),\ 0 \le j \le q-1. \tag{3.29}$$

Since $P_\epsilon(g^o)$ is an odd polynomial in Π^{2q-1}, we can expand $P_\epsilon(g^o)$ in terms of the basis $\{(x^2-1)^j x\}_{j=0}^{q-1}$. Therefore, from (3.29) and Lemma 2.3 it follows that $xP_\epsilon(g^o) \to 0$, as $\epsilon \to 0$, i.e., $P_0(g^o) = 0$. □

Now, we establish the second main result.

Theorem 3.11. *Let $n = 2q-1$, $f \in \tau_{q-1}(\pm 1)$ and $f^o \in t^2_{q-2}(1)$. Then there exists $P_0(f)$, and it is determined by the conditions $P_0^{(j)}(\pm 1) = f^{(j)}(\pm 1)$, $0 \le j \le q-1$.*

Proof. Since $f^o \in t^2_{q-2}(1)$, then $g^o \in t^2_{q-2}(1)$. In consequence, by Proposition 3.10 we get $P_0(g) = 0$. Finally, the theorem follows from Remark 3.8. □

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