MULTIVALUED EXTENDED BEST Φ–POLYNOMIAL APPROXIMATION OPERATOR

SONIA ACINAS[1] AND SERGIO FAVIER[2]

Abstract. In this paper we consider the best multivalued polynomial approximation operator defined in an Orlicz space $L^\Phi(B)$, where Φ is not necessarily an N-function. We also deal with the extended operator to $L^\varphi(B)$, being φ the derivative of Φ. Thus, the extension of the best polynomial approximation operator from $L^1(B)$ to $L^0(B)$ arises as a particular case of this work taking $\Phi(x) = x$.

Keywords and Phrases.
Orlicz Spaces, Best Polynomial Φ-Approximation Operators, Multivalued Extended Best Polynomial Approximation.
2010 *Mathematical Subject Classification.*
Primary 41A10. Secondary 41A50, 46E30.

1. Introduction

In this paper we set $\Im$ for the class of all continuous and nondecreasing functions φ defined for all real number $t \geq 0$, such that $\varphi(t) > 0$ for $t > 0$.

We also assume a Δ_2 condition for the functions φ, which means that there exists a constant $\Lambda = \Lambda_\varphi > 0$ such that $\varphi(2a) \leq \Lambda_\varphi \varphi(a)$ for all $a \geq 0$.

Now given $\varphi \in \Im$ we consider $\Phi(x) = \int_0^x \varphi(t)\,dt$. Observe that $\Phi : [0,\infty) \to [0,\infty)$ is a convex function such that $\Phi(a) = 0$ iff $a = 0$.

In the case that $\frac{\Phi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\Phi(x)}{x} \to \infty$ as $x \to \infty$, and according to [9], Φ is an N-function and its derivative function φ satisfies $\varphi(0^+) = 0$ and $\varphi(x) \to \infty$ as $x \to \infty$.

Observe that the function φ satisfies a Δ_2 condition if and only if the function Φ satisfies a Δ_2 condition, and since $\varphi \in \Im$ satisfies a Δ_2 condition, the next inequality holds

$$\frac{1}{2}(\varphi(a) + \varphi(b)) \leq \varphi(a + b) \leq \Lambda_\varphi(\varphi(a) + \varphi(b)), \tag{1.1}$$

for every $a, b \geq 0$.

Also note that the Δ_2 condition on Φ implies

$$\frac{x}{2\Lambda_\varphi}\varphi(x) \leq \Phi(x) \leq x\varphi(x), \tag{1.2}$$

for every $x \geq 0$.

Let B be a bounded measurable set in $\mathbb{R}^n$. If $\varphi \in \Im$, we denote by $L^\varphi(B)$ the class of all Lebesgue measurable functions f defined on B such that $\int_B \varphi(|f|)\,dx < \infty$. For the convex function Φ, $L^\Phi(B)$ is the classical Orlicz space very well studied in [9] and [14].

This paper was supported by CONICET and UNSL grants.

Let Π^m be the space of algebraic polynomials, defined on $\mathbb{R}^n$, of degree at most m. A polynomial $P \in \Pi^m$ is said to be a best approximation of $f \in L^\Phi(B)$ if and only if

$$\int_B \Phi(|f-P|)\,dx = \inf_{Q\in\Pi^m} \int_B \Phi(|f-Q|)\,dx. \tag{1.3}$$

Definition 1. *For $f \in L^\Phi(B)$ we denote by $\mu_\Phi(f)$ the set of all polynomials P that satisfy (1.3).*

Hereinafter we also refer to $\mu_\Phi(f)$ as the multivalued operator defined for functions in $L^\Phi(B)$ and images on Π^m.

In [1] existence and extension of the best polynomial approximation operator $\mu_\Phi(f)$ are stated assuming that the function Φ is a differentiable N-function, i.e. a differentiable convex function in $\Im$ such that its derivative function φ satisfy $\varphi(0^+) = 0$ and $\varphi(x) \to \infty$ as $x \to \infty$. The main goal of this paper is to obtain the extension of $\mu_\Phi(f)$ when Φ is not an N-function. We consider this extension in Theorem 3.3 on the assumption that φ is an unbounded function and $\varphi(0^+) > 0$; while, in Theorem 3.5 we study the problem when φ is an upper bounded function and $\varphi(0^+) \geq 0$. Thus, in Theorem 3.5 we generalize the case $\Phi(x) = x$ for functions in a bigger space than the one treated in [4] by Cuenya. We point out that the techniques employed to prove these results differ from that used in Theorem 3.3 of [1] due to the different behaviour of the function φ at 0 or at ∞.

The classic example of an extension of the best approximation operator is the conditional expectation. We can think about the conditional expectation as the projection of a function $f \in L^2$ on a probability space $(\Omega, \mathcal{A}, P)$ onto the subspace of $\mathcal{A}_0$ measurable functions which are in L^2 and where $\mathcal{A}_0$ denotes a sub sigma-algebra of $\mathcal{A}$. Since this projection is a monotone operator we can extend this best approximation operator from L^2 to L^1 and thus we obtain the well known conditional expectation operator. For $\Phi(x) = x^p,\ 1 < p < \infty$, a similar best approximation operator is considered in L^p and then extended to L^{p-1}, see [11]. In that paper the approximation class is the set of all the $\mathcal{A}_0$ measurable functions in L^p, where now $\mathcal{A}_0$ is a sub-sigma lattice of $\mathcal{A}$. This best approximation operator and other classical operators in harmonic analysis are also studied in a general Orlicz spaces L^Φ, see [10], [2] or [8]. If $\mathcal{A}_0 = \{\emptyset, \Omega\}$, that is, when the approximation class is the set of constant functions in Ω, the extension of the best approximation operator is detailed discussed in several papers, see [12], [6] and [7]. Also the extension of the best approximation operator in L^Φ for a general sub sigma-lattice $\mathcal{A}_0$ is treated in [3]. In all these cases it is strongly used the monotonicity of the best approximation operator in the space where it is originally defined.

If the approximation class is the algebraic polynomials we lose the monotonicity of the best approximation operator and the extension has to be tackled in a different way. For the L^2 case see [12] and for the L^p case we refer to [4]. In Orlicz spaces L^Φ, as we have said at the beginning, the best polynomial approximation operator and its extension are studied in [1], where it is required the function Φ to be an N-function.

We devote Section 2 to adjust Theorem 2.2, Theorem 2.3 and Theorem 2.4 of [1] for functions $f \in L^\Phi(B)$ where Φ is not an N-function. Although their related proofs follow the same way than in [1], we include them to make this paper self contained. In Section 3, we extend the best polynomial approximation operator from $L^\Phi(B)$ to $L^\varphi(B)$ where $\varphi = \Phi'$ developing a different method to that used in [1], since in this article Φ is not an N-function.

2. The best polynomial approximation operator in $L^\Phi(B)$

For $P \in \Pi^m$ we set $\|P\|_\infty = \max_{x\in B} |P(x)|$ and $\|P\|_1 = \int_B |P|\, dx$.

First, we prove that the set $\mu_\Phi(f)$ is not empty for functions $f \in L^\Phi(B)$ without assuming that $\varphi(0^+) = 0$ and $\varphi(x) \to \infty$ as $x \to \infty$.

Theorem 2.1. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\, dt$ and let $f \in L^\Phi(B)$. Then, there exists $P \in \Pi^m$ such that*

$$\int_B \Phi(|f - P|)\, dx = \inf_{Q\in\Pi^m} \int_B \Phi(|f - Q|)\, dx.$$

Proof. Let $I = \inf_{Q\in\Pi^m} \int_B \Phi(|f - Q|)\, dx$, then there exists a sequence $\{P_n\}_{n\in\mathbb{N}} \subset \Pi^m$ such that

$$\int_B \Phi(|f - P_n|)\, dx \to I \text{ as } n \to \infty.$$

Due to the monotonicity and convexity of Φ on $[0, \infty)$, we get

$$\Phi\left(\frac{|P_n|}{2}\right) \le \Phi\left(\frac{1}{2}|P_n - f| + \frac{|f|}{2}\right) \le \frac{1}{2}\Phi(|P_n - f|) + \frac{1}{2}\Phi(|f|).$$

Thus

$$\int_B \Phi\left(\frac{|P_n|}{2}\right) dx \le \frac{1}{2}\int_B \Phi(|P_n - f|)\, dx + \frac{1}{2}\int_B \Phi(|f|)\, dx,$$

and then

$$2\int_B \Phi\left(\frac{|P_n|}{2}\right) dx \le \int_B \Phi(|f|)\, dx + I + 1.$$

Now, Lemma 2.1 of [1] implies $\|P_n\|_\infty \le K$. Hence, there exists a subsequence $\{P_{n_k}\} \subseteq \{P_n\}_{\{n\in\mathbb{N}\}}$ such that $\{P_{n_k}\}$ converges uniformly on Π^m. Let $P = \lim_{n_k\to\infty} P_{n_k}$. Since Φ satisfies a Δ_2 condition we have

$$\Phi(|f - P_{n_k}|) \le \Lambda_\Phi(\Phi(|f|) + \Phi(|P_{n_k}|)) \le \Lambda_\Phi(\Phi(|f|) + \Phi(K)).$$

Therefore, by Lebesgue Dominated Convergence Theorem, we have $I = \int_B \Phi(|f - P|)\, dx$. □

Theorem 2.3 of [1] gives a characterization of the best polynomial approximation of functions in $L^\Phi(B)$ for $\varphi \in \Im$ such that $\varphi(0^+) = 0$ and $\varphi(x) \to \infty$ as $x \to \infty$. Next, proceeding in a similar way, we obtain a characterization of $\mu_\Phi(f)$ assuming only $\varphi(0^+) \ge 0$, which includes the L^1 case considered in [13].

Theorem 2.2. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let $f \in L^{\Phi}(B)$. If $\varphi(0^+) \geq 0$, then $P \in \Pi^m$ is in $\mu_\Phi(f)$ if and only if*

$$\left|\int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx\right| \leq \varphi(0^+)\int_{\{f=P\}} |Q|\,dx, \tag{2.1}$$

for any $Q \in \Pi^m$.

Proof. For P in $\mu_\Phi(f)$ and $Q \in \Pi^m$ we set

$$F_Q(\varepsilon) = \int_B \Phi(|f-P+\varepsilon Q|)\,dx.$$

It is easy to see that F_Q is a non negative convex function defined on $[0,\infty)$ such that

$$F_Q(0) = \min_{[0,\infty)} F_Q(\epsilon),$$

and this identity holds if and only if $0 \leq F'_Q(0^+)$.
Let

$$F_Q(\varepsilon) = \int_{B-\{f=P\}} \Phi(|f-P+\varepsilon Q|)\,dx + \int_{\{f=P\}} \Phi(\varepsilon|Q|)\,dx. \tag{2.2}$$

On $B - \{f-P=0\}$, by Mean Value Theorem, we get

$$\frac{|\Phi(|f-P+\epsilon Q|) - \Phi(|f-P|)|}{\epsilon|Q|} \leq |Q|\Lambda_\varphi(\varphi(|f-P|)+\varphi(|Q|)),$$

for $0 \leq \epsilon \leq 1$; and, on $\{f-P=0\}$ we have

$$\frac{|\Phi(\epsilon|Q|) - \Phi(0)|}{\epsilon|Q|} \leq \varphi(\delta|Q|)|Q| \leq \varphi(|Q|)|Q|,$$

for any $0 \leq \delta \leq \epsilon \leq 1$.
Then, since $|Q|\Lambda_\varphi(\varphi(|f-P|)+\varphi(|Q|))$ and $\varphi(|Q|)|Q|$ are integrable functions, we are allowed to differentiate inside the integral in the formula of $F_Q(\epsilon)$ given by (2.2) and therefore, using the fact that $\varphi(0^+) \geq 0$, we obtain

$$\begin{aligned} &0 \leq F'_Q(0^+) = \\ &\int_{B-\{f=P\}} \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx + \varphi(0^+)\int_{\{f=P\}} |Q|\,dx = \\ &\int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx + \varphi(0^+)\int_{\{f=P\}} |Q|\,dx, \end{aligned}$$

for any $Q \in \Pi^m$. Then,

$$-\int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx \leq \varphi(0^+)\int_{\{f=P\}} |Q|\,dx, \tag{2.3}$$

for any $Q \in \Pi^m$. Now for any polynomial $Q \in \Pi^m$ we take the polynomial $-Q$ in (2.3) and we get the wished inequality (2.1). □

Remark 2.3. Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let $f \in L^{\Phi}(B)$. If $\varphi(0^+) = 0$, then $P \in \Pi^m$ is in $\mu_\Phi(f)$ if and only if

$$\int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx = 0, \tag{2.4}$$

for every $Q \in \Pi^m$.

We point out that (2.4) is obtained without asking φ to be unbounded, while such a condition was required in Theorem 2.3 of [1].

The following result, in the spirit of Theorem 2.1 of [5] and Theorem 2.4 of [1], gives us an inequality which will become an useful tool.

Theorem 2.4. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let $f \in L^\varphi(B)$. Suppose the polynomial $P \in \Pi^m$ satisfies*

$$\left|\int_B \varphi(|f-P|)\text{sgn}(f-P)Q\,dx\right| \le K \int_{\{f=P\}} |Q|\,dx, \tag{2.5}$$

with $K \ge 0$ and for every $Q \in \Pi^m$. Then

$$\int_B \varphi(|P|)|Q|\,dx \le 5\Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx + K\int_{\{f=P\}} |Q|\,dx, \tag{2.6}$$

for every $Q \in \Pi^m$ satisfying $\text{sgn}(Q(t)P(t)) = (-1)^\eta$ *at any $t \in B$ such that $Q(t)P(t) \neq 0$ and where $\eta = 0$ or $\eta = 1$.*

Proof. First, suppose $Q \in \Pi^m$ such that $Q(x)P(x) > 0$.
Let $N = \{x \in B : f(x) > P(x)\}$ and $L = \{x \in B : f(x) \le P(x)\}$.
Then, from (2.5), we have

$$-K\int_{\{f=P\}} |Q|\,dx \le \int_{N\cup L} \varphi(|f-P|)\,\text{sgn}(f-P)Q\,dx =$$
$$\int_N \varphi(|f-P|)\,Q\,dx - \int_L \varphi(|f-P|)\,Q\,dx \le K\int_{\{f=P\}} |Q|\,dx.$$

Thus

$$\begin{aligned} &-K\int_{\{f=P\}} |Q|\,dx - \int_N \varphi(|f-P|)\,Q\,dx + \int_L \varphi(|f-P|)\,Q\,dx \le \\ &0 \le \\ &K\int_{\{f=P\}} |Q|\,dx - \int_N \varphi(|f-P|)\,Q\,dx + \int_L \varphi(|f-P|)\,Q\,dx. \end{aligned} \tag{2.7}$$

Let $H(x) = \varphi(|P(x) - f(x)|)Q(x)$ and consider the sets
$U_1 = N \cap \{x \in B : P(x) \ge 0\}$, $U_2 = N \cap \{x \in B : P(x) < 0\}$,
$U_3 = L \cap \{x \in B : P(x) \ge 0\}$, $U_4 = L \cap \{x \in B : P(x) < 0\}$.
Then, by the first inequality in (2.7), we get

$$\int_{U_3\cup U_4} H\,dx \le K\int_{\{f=P\}} |Q|\,dx + \int_{U_1\cup U_2} H\,dx,$$

and therefore

$$\int_{U_3} H\,dx - \int_{U_2} H\,dx \le K\int_{\{f=P\}} |Q|\,dx + \int_{U_1} H\,dx - \int_{U_4} H\,dx. \tag{2.8}$$

By similar calculation to the one given in the proof of Theorem 2.4 of [1], we obtain

$$\int_B \varphi(|P|)|Q|\,dx \le \Lambda_\varphi \int_B \varphi(|P-f|)|Q|\,dx + \Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx =$$
$$\Lambda_\varphi \int_{\bigcup_{i=1}^4 U_i} |H|\,dx + \Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx =$$
$$\Lambda_\varphi \sum_{i=1}^4 \int_{U_i} |H|\,dx + \Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx = \Lambda_\varphi(I_1+I_2),$$

and

$$\int_{U_1\cup U_4} |H|\,dx \le 2\int_B \varphi(|f|)|Q|\,dx. \tag{2.9}$$

Since $\operatorname{sgn}Q = \operatorname{sgn}P$, from (2.8) and (2.9), we get

$$\begin{aligned}&\int_{U_2} |H|\,dx + \int_{U_3} |H|\,dx = \int_{U_2} (-H)\,dx + \int_{U_3} H\,dx \le\\ &K\int_{\{f=P\}} |Q|\,dx + \int_{U_1} H\,dx - \int_{U_4} H\,dx =\\ &K\int_{\{f=P\}} |Q|\,dx + \int_{U_1} |H|\,dx + \int_{U_4} |H|\,dx \le\\ &K\int_{\{f=P\}} |Q|\,dx + 2\int_B \varphi(|f|)|Q|\,dx.\end{aligned} \tag{2.10}$$

Therefore $I_1 \le 4\int_B \varphi(|f|)|Q|\,dx$ and

$$\int_B \varphi(|P|)|Q|\,dx \le 5\Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx + K\int_{\{f=P\}} |Q|\,dx. \tag{2.11}$$

Now if $Q \in \Pi^m$ satisfies $Q(x)P(x) < 0$ we proceed in an analogous way to obtain (2.10). By the second inequality in (2.7), we have

$$\int_{U_1\cup U_2} H\,dx \le K\int_{\{f=P\}} |Q|\,dx + \int_{U_3\cup U_4} H\,dx,$$

and thus

$$\int_{U_1} H\,dx - \int_{U_4} H\,dx \le K\int_{\{f=P\}} |Q|\,dx + \int_{U_3} H\,dx - \int_{U_2} H\,dx;$$

then

$$\int_{U_2} |H|\,dx + \int_{U_3} |H|\,dx = \int_{U_2} H\,dx - \int_{U_3} H\,dx \le$$
$$K\int_{\{f=P\}} |Q|\,dx - \int_{U_1} H\,dx + \int_{U_4} H\,dx =$$
$$K\int_{\{f=P\}} |Q|\,dx + \int_{U_1} |H|\,dx + \int_{U_4} |H|\,dx =$$
$$K\int_{\{f=P\}} |Q|\,dx + \int_{U_1\cup U_4} |H|\,dx \le 2\int_B \varphi(|f|)|Q|\,dx + K\int_{\{f=P\}} |Q|\,dx,$$

and thus

$$\int_B \varphi(|P|)|Q|\,dx \le 5\Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx + K\int_{\{f=P\}} |Q|\,dx, \tag{2.12}$$

for $Q \in \Pi^m$ such that $Q(x)P(x) < 0$.
Finally, (2.6) follows from (2.11) and (2.12) □

Corollary 2.5. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let $f \in L^\Phi(B)$.*
If $\varphi(0^+) > 0$ and P is the best polynomial approximation of $f \in L^\Phi(B)$, then

$$\int_B \varphi(|P|)|P|\,dx \le 5\Lambda_\varphi \|P\|_\infty \int_B \varphi(|f|)\,dx + \varphi(0^+)|B|\|P\|_\infty = \tilde{K}\|P\|_\infty. \tag{2.13}$$

Proof. It follows from Theorem 2.4 with $Q = P$ and $K = \varphi(0^+) > 0$, employing the fact that $|P| \le \|P\|_\infty$ and taking $\tilde{K} = 5\Lambda_\varphi \int_B \varphi(|f|)\,dx + \varphi(0^+)|B|$. □

Remark 2.6. In order to obtain Theorem 2.4 we only have used that the polynomial P is a solution of (2.5) regardless of the space to which f belongs. Thus the inequality (2.13) holds for any polynomial P that satisfies the inequality (2.5) and f belonging to $L^\varphi(B)$.

3. Extension of the best polynomial approximation to $L^\varphi(B)$

In Theorem 3.3 of [1] we have proved that $\mu_\varphi(f)$ is a non empty set and then we could extend the operator $\mu_\Phi(f)$ from the space $L^\Phi(B)$ to a bigger space $L^\varphi(B)$ for an N-function Φ given by $\Phi(x) = \int_0^x \varphi(t)\,dt$. In this section, we will get an extension of $\mu_\Phi(f)$ without requesting the function φ to satisfy $\varphi(0^+) = 0$ or $\varphi(x) \to \infty$ as $x \to \infty$ as follows.

Definition 2. *Let $\varphi \in \Im$.*
For $f \in L^\varphi(B)$ we denote by $\mu_\varphi(f)$ the set of polynomials $P \in \Pi^m$ that satisfy

$$\left|\int_B \varphi(|f-P|)\operatorname{sgn}(f-P)Q\,dx\right| \le \varphi(0^+)\int_{\{f=P\}} |Q|\,dx, \tag{3.1}$$

for every $Q \in \Pi^m$; and, we refer to this set as the extended best polinomial approximation operator.

Before going straight to the point, we state an auxiliary result that will be used later.

Lemma 3.1. *Let $\varphi \in \Im$ be a function that satisfies $0 \le \varphi(0^+) \le \varphi(\infty)$ where $\varphi(\infty) = \lim_{x\to\infty} \varphi(x)$.*
Suppose that φ_n is a sequence of functions in $\Im$ such that φ_n uniformly converges to φ as $n \to \infty$ on $[\delta, \infty)$ for every $\delta > 0$.

1. *If x_n is a sequence of real positive numbers such that $x_n \to \infty$ when $n \to \infty$, then $\varphi_n(x_n) \to \varphi(\infty)$ as $n \to \infty$.*
2. *If x_n is a sequence of real positive numbers such that $x_n \to x$ when $n \to \infty$ and where $x > 0$, then $\varphi_n(x_n) \to \varphi(x)$ as $n \to \infty$.*

Proof. It follows straightforwardly from the uniform convergence of the sequence of functions φ_n to the function φ as $n \to \infty$ on $[\delta, \infty)$. ☐

First, we will consider the case where $\varphi \in \Im$ is an upper unbounded function and $\varphi(0^+) > 0$.

We begin with an application of Corollary 2.5 of [1] to get uniform boundedness of a set of polynomials.

Lemma 3.2. *Let $\varphi \in \Im$ be an upper unbounded function such that $\varphi(0^+) > 0$. Let $\varphi_n \in \Im$ be a sequence of functions such that φ_n uniformly converges to φ as $n \to \infty$ $[\delta, \infty)$ for every $\delta > 0$, $\varphi_n(0^+) = 0$ for every $n \in \mathbb{N}$ and $\varphi_n(x) \leq \varphi(x)$ for every $x \geq 0$ and for every $n \in \mathbb{N}$. Moreover, assume that the sequence Λ_{φ_n} is bounded.*
Let $f_n \in L^{\Phi_n}(B)$ for every $n \in \mathbb{N}$ where $\Phi_n(x) = \int_0^x \varphi_n(t)\, dt$.
If there exists a constant C that satisfies $\int_B \varphi_n(|f_n|)\, dx \leq C$, then $\{\|P_n\|_\infty : P_n \in \mu_{\Phi_n}(f_n), n = 1, 2, \ldots\}$ is bounded.

Proof. Let $P_n \in \mu_{\Phi_n}(f_n)$ for every $n \in \mathbb{N}$. By Remark 2.3, we have

$$0 = \int_B \varphi_n(|f_n - P_n|)\text{sgn}(f_n - P_n)Q\, dx \;\text{ for every }\; Q \in \Pi^m.$$

Then, using Corollary 2.5 in [1], we have

$$\int_B \varphi_n(|P_n|)|P_n|\, dx \leq 5\Lambda_{\varphi_n}\|P_n\|_\infty \int_B \varphi_n(|f_n|)\, dx \leq 5C\Lambda_{\varphi_n}\|P_n\|_\infty,$$

for every $n \in \mathbb{N}$. From here, the proof follows as in Lemma 3.1 of [1] employing (1.2), Jensen's inequality and (2.14) of Lemma 2.4 in [5]. In this way, we get

$$\Phi_n\left(\frac{C_1}{|B|}\|P_n\|_\infty\right) \leq \frac{5C\Lambda_{\varphi_n}}{C_1}\left(\frac{C_1\|P_n\|_\infty}{|B|}\right),$$

where the constant C_1 comes from the equivalence between $\|\cdot\|_\infty$ and $\|\cdot\|_1$ for polynomials (Lemma 2.4 in [5]) and this constant depends only on the dimension n and the degree m. Now, by (1.2) and the fact that there exists $K > 0$ such that $\Lambda_{\varphi_n} \leq K$ for every $n \in \mathbb{N}$, we obtain

$$\varphi_n\left(\frac{C_1}{|B|}\|P_n\|_\infty\right) \leq \tilde{K} \;\text{ for every } n \in \mathbb{N}, \tag{3.2}$$

being $\tilde{K} = \frac{10CK^2}{C_1}$ independent of φ_n and P_n.
By Lemma 3.1 (1), we know that $\varphi_n(x_n) \to \infty$ as $x_n \to \infty$; consequently, (3.2) implies that $\{\|P_n\|_\infty\}$ is uniformly bounded. ☐

Now, we prove that $\mu_\varphi(f) \neq \emptyset$.

Theorem 3.3. *Let $\varphi \in \Im$ be an upper unbounded function such that $\varphi(0^+) > 0$.*
If $f \in L^\varphi(B)$, then there exists $P \in \Pi^m$ such that

$$\left|\int_B \varphi(|f - P|)\text{sgn}(f - P)Q\, dx\right| \leq \varphi(0^+)\int_{\{f=P\}} |Q|\, dx, \tag{3.3}$$

for every $Q \in \Pi^m$.
And

$$\int_B \Phi(|P|)\,dx \le C\|P\|_\infty \left(\int_B \varphi(|f|)\,dx + 1\right),$$

for a suitable constant C.

Proof. Let $\{\varphi_n\}$ be a sequence of functions in $\Im$ such that $\varphi_n \xrightarrow{u} \varphi$ as $n \to \infty$ on $[\delta, \infty)$ for every $\delta > 0$, $\varphi_n(0^+) = 0$ for every $n \in \mathbb{N}$ and $\varphi_n(x) \le \varphi(x)$ for every $x \ge 0$ and for every $n \in \mathbb{N}$. We also assume that the sequence Λ_{φ_n} is bounded.
Let $f \in L^\varphi(B)$ and we define $f_n = \min(\max(f, -n), n)$, then $|f_n| \le |f|$ for every $n \in \mathbb{N}$, $f_n \to f$ as $n \to \infty$ and $f_n \in L^{\Phi_n}(B)$ for every $n \in \mathbb{N}$ being $\Phi_n(x) = \int_0^x \varphi_n(t)\,dt$.
As $|f_n| \le |f|$ and $f \in L^\varphi(B)$, there exists $C > 0$ such that $\int_B \varphi_n(|f_n|)\,dx \le C$. Then, by Lemma 3.2, there exists $M > 0$ such that $\|P_n\|_\infty \le M$ for every $n \in \mathbb{N}$, being $P_n \in \mu_{\Phi_n}(f_n)$ for each $n \in \mathbb{N}$. Consequently, there exists a subsequence P_{n_k} which uniformly converges on B to a polynomial $P \in \Pi^m$. As $P_{n_k} \in \mu_\Phi(f_{n_k})$, by Remark 2.3, we have

$$0 = \int_B \varphi_{n_k}(|f_{n_k} - P_{n_k}|)\mathrm{sgn}(f_{n_k} - P_{n_k})Q\,dx \quad \text{for every} \quad Q \in \Pi^m.$$

Now,

$$\begin{aligned}
&\left|\int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx\right| \le \\
&\left|\int_{B-\{f=P\}} \varphi_{n_k}(|f_{n_k} - P_{n_k}|)\mathrm{sgn}(f_{n_k} - P_{n_k})Q\,dx - \right. \\
&\left.\int_{B-\{f=P\}} \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx\right| + \\
&\int_{\{f=P\}} \varphi_{n_k}(|f_{n_k} - P_{n_k}|)|\mathrm{sgn}(f_{n_k} - P_{n_k})||Q|\,dx,
\end{aligned} \tag{3.4}$$

for every $Q \in \Pi^m$.
As $P_{n_k} \xrightarrow{u} P$ and $f_{n_k} \to f$ on B when $k \to \infty$ and $\varphi_n \xrightarrow{u} \varphi$ on $[\delta, \infty)$ for every $\delta > 0$, then, from Lemma 3.1 (2), $\varphi_{n_k}(|f_{n_k} - P_{n_k}|) \to \varphi(|f-P|)$ on $B - \{f = P\}$ as $k \to \infty$. In addition, $\mathrm{sgn}(f_{n_k} - P_{n_k}) \to \mathrm{sgn}(f - P)$ on $B - \{f = P\}$.
Since $|\mathrm{sgn}(f_{n_k} - P_{n_k})| \le 1$ for every $k \in \mathbb{N}$, $0 \le \varphi_n(x) \le \varphi(x)$ for every $n \in \mathbb{N}$ and for every $x \ge 0$, and $\varphi \in \Im$ satisfies a Δ_2 condition, we get

$$\begin{aligned}
&\varphi_{n_k}(|f_{n_k} - P_{n_k}|)|\mathrm{sgn}(f_{n_k} - P_{n_k})||Q| \le \\
&\varphi(|f_{n_k} - P_{n_k}|)|Q| \le \\
&\Lambda_\varphi[\varphi(|f_{n_k}|) + \varphi(|P_{n_k}|)]|Q| \le \\
&\Lambda_\varphi[\varphi(|f|) + \varphi(M)]|Q| \in L^1(B),
\end{aligned} \tag{3.5}$$

for every $k \in \mathbb{N}$ and for every $Q \in \Pi^m$.
Thus, by Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}(3.6)\qquad &\int_{B-\{f=P\}} \varphi_{n_k}(|f_{n_k}-P_{n_k}|)\mathrm{sgn}(f_{n_k}-P_{n_k})Q\,dx \to \\ &\int_{B-\{f=P\}} \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx,\end{aligned}$$

as $k \to \infty$ and for every $Q \in \Pi^m$.
On the other hand, by (3.5), we have

$$\begin{aligned}(3.7)\qquad &\int_{\{f=P\}} \varphi_{n_k}(|f_{n_k}-P_{n_k}|)|\mathrm{sgn}(f_{n_k}-P_{n_k})||Q|\,dx \le \\ &\int_{\{f=P\}} \varphi(|f_{n_k}-P_{n_k}|)|Q|\,dx,\end{aligned}$$

for every $Q \in \Pi^m$.
Also, $\varphi(|f_{n_k}-P_{n_k}|)|Q| \to \varphi(0^+)|Q|$ as $k \to \infty$ on $\{f = P\}$ and, reasoning as it has done to obtain (3.5), we have

$$\varphi(|f_{n_k}-P_{n_k}|)|Q| \le \Lambda_\varphi[\varphi(|f|)+\varphi(|M|)]|Q| \in L^1(B),$$

for every $k \in \mathbb{N}$ and for every $Q \in \Pi^m$.
Then, from Lebesgue Dominated Convergence Theorem, we obtain

$$(3.8)\qquad \int_{\{f=P\}} \varphi(|f_{n_k}-P_{n_k}|)|Q|\,dx \to \varphi(0^+)\int_{\{f=P\}} |Q|\,dx,$$

as $k \to \infty$ and for every $Q \in \Pi^m$. Therefore, from (3.4), (3.6), (3.7) and (3.8), we have

$$\left|\int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx\right| \le \varphi(0^+)\int_{\{f=P\}} |Q|\,dx,$$

for every $Q \in \Pi^m$. Now, by Remark 2.6 and (1.2), we get

$$\int_B \Phi(|P|)\,dx \le \int_B \varphi(|P|)|P|\,dx \le C\|P\|_\infty \left(\int_B \varphi(|f|)\,dx + 1\right),$$

with $C = \max\{5\Lambda_\varphi, \tilde{K}\}$, being $\tilde{K}$ a constant depending on $|B|$, $\varphi(0^+)$ and another constant coming from the equivalence between $\|\cdot\|_1$ and $\|\cdot\|_\infty$ for polynomials, and the proof is completed. □

Next, we deal with the case where $\varphi \in \Im$ is an upper bounded function and $\varphi(0^+) \ge 0$.

Let $\{\varphi_n\}$ be a sequence of functions in $\Im$ such that φ_n uniformly converges to φ as $n \to \infty$ on $[\delta, \infty)$ for every $\delta > 0$, $\varphi_n(0^+) = 0$ for every $n \in \mathbb{N}$ and $\varphi_n(x) \le \varphi(x)$ for every $x \ge 0$ and for every $n \in \mathbb{N}$.

Let $f \in L^\varphi(B)$ and we define $f_n = \min(\max(f, -n), n)$, then $|f_n| \le |f|$ for every $n \in \mathbb{N}$, $f_n \to f$ as $n \to \infty$ and $f_n \in L^{\Phi_n}(B)$ for every $n \in \mathbb{N}$ with $\Phi_n(x) = \int_0^x \varphi_n(t)\,dt$.

Let $P_n \in \mu_{\Phi_n}(f_n)$ for each $n \in \mathbb{N}$. Then, by Remark 2.3, we have

$$(3.9)\qquad 0 = \int_B \varphi_n(|f_n-P_n|)\mathrm{sgn}(f_n-P_n)Q\,dx \ \text{ for every } \ Q \in \Pi^m.$$

We will see that the sequence $\{P_n\}$ is uniformly bounded.

Lemma 3.4. *Let $f_n \in L^{\Phi_n}(B)$ for every $n \in \mathbb{N}$. If $P_n \in \mu_{\Phi_n}(f_n)$ for each $n \in \mathbb{N}$, then $\{P_n\}$ is uniformly bounded.*

Proof. Suppose that $\{P_n\}$ is not uniformly bounded, then there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that $b_k = \|P_{n_k}\|_\infty \to \infty$ as $k \to \infty$.
Let $R_{n_k} = \frac{P_{n_k}}{b_k}$, then $\|R_{n_k}\|_\infty = 1$ for every $k \in \mathbb{N}$. Therefore, there exist a subsequence $\{R_{n_{k_j}}\} \subset \{R_{n_k}\}$ and a polynomial $R_0 \in \Pi^m$ such that $R_{n_{k_j}} \xrightarrow{u} R_0$ on B as $j \to \infty$ and $\|R_0\|_\infty = 1$. Let $Z = \{x \in B : R_0(x) = 0\}$, then $|Z| = 0$.
From (3.9) and as $b_k > 0$ for every $k \in \mathbb{N}$, for every $j \in \mathbb{N}$ we have

$$0 = \int_B \varphi_{n_{k_j}} \left(\left| \frac{f_{n_{k_j}}}{b_{n_{k_j}}} - R_{n_{k_j}} \right| b_{k_j} \right) \operatorname{sgn} \left(\frac{f_{n_{k_j}}}{b_{k_j}} - R_{n_{k_j}} \right) Q \, dx \ \text{ for every } \ Q \in \Pi^m.$$

As $\frac{f_{n_{k_j}}(x)}{b_{n_{k_j}}} - R_{n_{k_j}}(x) \to -R_0(x)$ when $j \to \infty$ at each $x \in B$, then

$$\operatorname{sgn} \left(\frac{f_{n_{k_j}}(x)}{b_{n_{k_j}}} - R_{n_{k_j}}(x) \right) \to \operatorname{sgn}(-R_0(x)), \tag{3.10}$$

when $j \to \infty$ at each $x \in B - Z$.
Let $K > 0$ be the upper bound of φ.
Now, since $\varphi_n \xrightarrow{u} \varphi$ on $[\delta, \infty)$ for every $\delta > 0$ and

$$\left| \frac{f_{n_{k_j}}(x)}{b_{n_{k_j}}} - R_{n_{k_j}}(x) \right| b_{k_j} \to \infty \ \text{ as } \ j \to \infty \ \text{ at each } \ x \in B - Z,$$

then, by Lemma 3.1 (1) and (3.10), we get

$$\varphi_{n_{k_j}} \left(\left| \frac{f_{n_{k_j}}(x)}{b_{n_{k_j}}} - R_{n_{k_j}}(x) \right| b_{k_j} \right) \operatorname{sgn} \left(\frac{f_{n_{k_j}}(x)}{b_{n_{k_j}}} - R_{n_{k_j}}(x) \right) Q \to$$
$$K \operatorname{sgn}(-R_0(x)) Q \ \text{ as } \ j \to \infty \ \text{ at each } \ x \in B - Z.$$

In addition, we have

$$\left| \varphi_{n_{k_j}} \left(\left| \frac{f_{n_{k_j}}(x)}{b_{n_{k_j}}} - R_{n_{k_j}}(x) \right| b_{k_j} \right) \operatorname{sgn} \left(\frac{f_{n_{k_j}}(x)}{b_{n_{k_j}}} - R_{n_{k_j}}(x) \right) Q \right| \leq$$
$$K|Q| \in L^1(B),$$

for every $x \geq 0$ and for every $n \in \mathbb{N}$.
Thus, by Lebesgue Dominated Convergence Theorem, we obtain

$$0 = \int_B K \operatorname{sgn}(-R_0) Q \, dx \ \text{ for every } \ Q \in \Pi^m.$$

We choose $Q = -R_0 \in \Pi^m$, then $\|R_0\|_1 = 0$ and therefore $R_0 \equiv 0 \in \Pi^m$, which contradicts the fact that $\|R_0\|_\infty = 1$. Consequenlty, the sequence $\{P_n\}$ is uniformly bounded, where $P_n \in \mu_{\varphi_n}(f_n)$ for each $n \in \mathbb{N}$. □

Now, we can prove that $\mu_\varphi(f) \neq \emptyset$.

Theorem 3.5. *Let $\varphi \in \Im$ be an upper bounded function such that $\varphi(0^+) \geq 0$. If $f \in L^{\varphi}(B)$, then there exists $P \in \Pi^m$ such that*

$$\left| \int_B \varphi(|f-P|)\operatorname{sgn}(f-P)Q\,dx \right| \leq \varphi(0^+) \int_{\{f=P\}} |Q|\,dx, \tag{3.11}$$

for every $Q \in \Pi^m$.
And

$$\int_B \Phi(|P|)\,dx \leq C\|P\|_\infty \left(\int_B \varphi(|f|)\,dx + 1 \right),$$

for a suitable constant C.

Proof. Let $\{\varphi_n\}$ be a sequence of functions in $\Im$ such that $\varphi_n \xrightarrow{u} \varphi$ as $n \to \infty$ on $[\delta, \infty)$ for every $\delta > 0$, $\varphi_n(0^+) = 0$ for every $n \in \mathbb{N}$ and $\varphi_n(x) \leq \varphi(x)$ for every $x \geq 0$ and for every $n \in \mathbb{N}$.
Let $f \in L^{\varphi}(B)$ and we define $f_n = \min(\max(f, -n), n)$, then $|f_n| \leq |f|$ for every $n \in \mathbb{N}$, $f_n \to f$ as $n \to \infty$ and $f_n \in L^{\Phi_n}(B)$ for every $n \in \mathbb{N}$ being $\Phi_n(x) = \int_0^x \varphi_n(t)\,dt$.
Lemma 3.4 implies that the sequence of polynomials $P_n \in \mu_{\Phi_n}(f_n)$ is uniformly bounded, then there exist a subsequence $\{P_{n_k}\} \subset \{P_n\}$ and a polynomial $P \in \Pi^m$ such that $P_{n_k} \xrightarrow{u} P$ as $k \to \infty$.
As $P_{n_k} \in \mu_{\Phi_n}(f_{n_k})$, from Remark 2.3, we have

$$0 = \int_B \varphi_{n_k}(|f_{n_k} - P_{n_k}|)\operatorname{sgn}(f_{n_k} - P_{n_k})Q\,dx \quad \text{for every} \quad Q \in \Pi^m.$$

Now,

$$\left| \int_B \varphi(|f-P|)\operatorname{sgn}(f-P)Q\,dx \right| \leq$$
$$\left| \int_{B-\{f=P\}} \varphi_{n_k}(|f_{n_k} - P_{n_k}|)\operatorname{sgn}(f_{n_k} - P_{n_k})Q\,dx - \right.$$
$$\left. \int_{B-\{f=P\}} \varphi(|f-P|)\operatorname{sgn}(f-P)Q\,dx \right| +$$
$$\int_{\{f=P\}} \varphi_{n_k}(|f_{n_k} - P_{n_k}|)|\operatorname{sgn}(f_{n_k} - P_{n_k})||Q|\,dx,$$

for every $Q \in \Pi^m$.
As $P_{n_k} \xrightarrow{u} P$ on B, $f_{n_k} \to f$ on B and $\varphi_n \xrightarrow{u} \varphi$ on $[\delta, \infty)$ for every $\delta > 0$, then, from Lemma 3.1 (2), $\varphi_{n_k}(|f_{n_k} - P_{n_k}|) \to \varphi(|f-P|)$ on $B - \{f = P\}$ as $k \to \infty$. And, we also have that $\operatorname{sgn}(f_{n_k} - P_{n_k}) \to \operatorname{sgn}(f-P)$ on $B - \{f = P\}$.
In addition, as $0 \leq \varphi_n(x) \leq \varphi(x)$ for every $n \in \mathbb{N}$ and for every $x \geq 0$ and $\varphi \in \Im$ is an upper bounded function, we get

$$\begin{aligned} &\varphi_{n_k}(|f_{n_k} - P_{n_k}|)|\operatorname{sgn}(f_{n_k} - P_{n_k})||Q| \leq \\ &\varphi(|f_{n_k} - P_{n_k}|)|\operatorname{sgn}(f_{n_k} - P_{n_k})||Q| \leq K|Q| \in L^1(B), \end{aligned} \tag{3.12}$$

for every $k \in \mathbb{N}$ and for every $Q \in \Pi^m$, where K is the upper bound of the function φ.

Thus, by Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}(3.13)\qquad &\int_{B-\{f=P\}} \varphi_{n_k}(|f_{n_k}-P_{n_k}|)\mathrm{sgn}(f_{n_k}-P_{n_k})Q\,dx \to \\ &\int_{B-\{f=P\}} \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx,\end{aligned}$$

as $k\to\infty$ and for every $Q\in\Pi^m$.
On the other hand, by (3.12) we get

$$\begin{aligned}(3.14)\qquad &\int_{\{f=P\}} \varphi_{n_k}(|f_{n_k}-P_{n_k}|)|\mathrm{sgn}(f_{n_k}-P_{n_k})||Q|\,dx \le \\ &\int_{\{f=P\}} \varphi(|f_{n_k}-P_{n_k}|)|Q|\,dx,\end{aligned}$$

for every $Q\in\Pi^m$.
We also have $\varphi(|f_{n_k}-P_{n_k}|)|Q| \to \varphi(0^+)|Q|$ as $k\to\infty$ on $\{f=P\}$ and, reasoning as in (3.12), we obtain $\varphi(|f_{n_k}-P_{n_k}|)|Q| \le K|Q| \in L^1(B)$ for every $k\in\mathbb{N}$ and for every $Q\in\Pi^m$.
Then, by Lebesgue Dominated Convergence Theorem, we get

$$(3.15)\qquad \int_{\{f=P\}} \varphi(|f_{n_k}-P_{n_k}|)|Q|\,dx \to \varphi(0^+)\int_{\{f=P\}}|Q|\,dx,$$

as $k\to\infty$ and for every $Q\in\Pi^m$.
Therefore, from (3.13), (3.14) and (3.15), we have

$$\left|\int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx\right| \le \varphi(0^+)\int_{\{f=P\}}|Q|\,dx,$$

for every $Q\in\Pi^m$. Now, by Remark 2.6 and (1.2), we also get

$$\int_B \Phi(|P|)\,dx \le \int_B \varphi(|P|)|P|\,dx \le C\|P\|_\infty\left(\int_B\varphi(|f|)\,dx+1\right),$$

with $C=\max\{5\Lambda_\varphi,\tilde K\}$, being $\tilde K$ a constant depending on $|B|$, $\varphi(0^+)$ and another constant coming from the equivalence between $\|\cdot\|_1$ and $\|\cdot\|_\infty$ for polynomials, and the proof is completed. □

If $\varphi(0^+)=0$, the inequality (3.1) becomes the identity (2.4) given in Remark 2.3; nevertheless, in (3.1) f belongs to the bigger set $L^\varphi(B)$. In this case, we can get uniqueness performing a similar procedure to that developed in Theorem 3.4 of [1] provided that φ is a strictly increasing function and without assuming that $\varphi(x)\to\infty$ as $x\to\infty$.

If $\varphi(x)\equiv 1$ on $[0,\infty)$, then the inequality (3.1) gives the extension of the best polynomial approximation operator from $L^1(B)$ to $L^0(B)$, being $L^0(B)$ the set of all measurable functions. This problem has been studied by Cuenya in [4] where the author obtained the extension for functions belonging to a proper subset of $L^0(B)$. Also, we point out that in Theorem 3.5 of this paper we obtain $\mu_\varphi(f)\neq\emptyset$ in a different way to that to that used in Theorem 5.4 of [4].

References

[1] S. Acinas, S. Favier, and F. Zó. Extended best polynomial approximation operator in Orlicz Spaces. *Numerical Functional Analysis and Optimization*, 36(7): 817–829, 2015.

[2] H.D. Brunk and S. Johansen. A generalized Radon-Nikodym derivative. *Pacific J. Math.*, 34:585–617, 1970.

[3] I. Carrizo, S. Favier, and F. Zó. Extension of the best approximation operator in Orlicz spaces. *Abstr. Appl. Anal.*, pages Art. ID 374742, 15, 2008.

[4] H. Cuenya. Extension of the operator of best polynomial approximation in $L^p(\Omega)$. *J. Math. Anal. Appl.*, 376(2):565–575, 2011.

[5] H. Cuenya, S. Favier, and F. Zó. Inequalities in L^{p-1} for the extended L^p best approximation operator. *J. Math. Anal. Appl.*, 393:80–88, 2012.

[6] S. Favier and F. Zó. A Lebesgue type differentiation theorem for best approximations by constants in Orlicz spaces. *Real Anal. Exchange*, 30(1):29–41, 2004-2005.

[7] S. Favier and F. Zó. Maximal inequalities for a best approximation operator in Orlicz spaces. *Comment. Math.*, 51(1):3–21, 2011.

[8] V. Kokilashvili and M. Krbec. *Weighted inequalities in Lorentz and Orlicz spaces.* World Scientific Publishing Co., Inc., River Edge, NJ, 1991.

[9] M.A. Krasnosel′skiĭ and Ja.B. Rutickiĭ. *Convex functions and Orlicz spaces.* P. Noordhoff Ltd., Groningen, 1961. Translated from the first Russian edition by Leo F. Boron.

[10] D. Landers and L. Rogge. Best approximants in L_Φ-spaces. *Z. Wahrsch. Verw. Gebiete*, 51(2):215–237, 1980.

[11] D. Landers and L. Rogge. Isotonic approximation in L_s. *J. Approx. Theory*, 31(3):199–223, 1981.

[12] F. Mazzone and H. Cuenya. Maximal inequalities and Lebesgue's differentiation theorem for best approximant by constant over balls. *J. Approx. Theory*, 110(2):171–179, 2001.

[13] A.M. Pinkus. *On L^1-approximation*, volume 93 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1989.

[14] M.M. Rao and Z.D. Ren. *Theory of Orlicz spaces.* M. Dekker, 1991.

[1] Instituto de Matemática Aplicada San Luis and CONICET, (5700) San Luis, Argentina. Departamento de Matemática, Facultad de Ciencias Exactas Naturales, Universidad Nacional de La Pampa, (6300) Santa Rosa, La Pampa, Argentina

E-mail address: sonia.acinas@gmail.com

[2] Instituto de Matemática Aplicada San Luis, CONICET and Departamento de Matemática, Universidad Nacional de San Luis, (5700) San Luis, Argentina. T.E. 54 266 4459649

E-mail address: sfavier@unsl.edu.ar