OPINION FORMATION MODELS WITH HETEROGENEOUS PERSUASION AND ZEALOTRY

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ABSTRACT. In this work an opinion formation model with heterogeneous agents is proposed. Each agent is supposed to have different power of persuasion, and besides its own level of zealotry, that is, an individual willingness to being convinced by other agent. In addition, our model includes zealots or stubborn agents, agents that never change opinions.

We derive a Bolzmann-like equation for the distribution of agents on the space of opinions, which is approximated by a transport equation with a nonlocal drift term. We study the long-time asymptotic behavior of solutions, characterizing the limit distribution of agents, which consists of the distribution of stubborn agents, plus a delta function at the mean of their opinions, weighted by they power of persuasion.

Moreover, explicit bounds on the rate of convergence are given, and the time to convergence is shown to decrease when the number of stubborn agents increases. This is a remarkable fact observed in agent based simulations in different works.

1. INTRODUCTION

In recent years, opinion formation, as well as other sociological and economical phenomena, have attracted a considerable attention from physicists and mathematicians, as it was realized that concepts from statistical mechanics could be successfully applied to model them. We refer for instance to the papers by S. Galam [26, 28], Sznajd and Sznajd-Veron [45], Deffuant et al. [16, 17], and Slanina [43] among other works. From them, quickly emerged two very active new fields, usually called *sociophysics* and *econophysics*, devoted to the description of these phenomena from the physicists point of view. We underline some recent books [13, 29, 42, 43] for an overview and up-to-date references.

In the sociophysics community, a customary procedure for modelling the formation of opinions in a population consists in representing the opinion of an individual, with respect to certain subject, by a real number. This number can vary in some discrete set or in a fixed interval, say $[-1, 1]$, meaning -1 to be completely against the subject. Individual changes of opinion are assumed to be a result of binary random interactions between agents. The opinions w and w_* of two agents will turn to new opinions w' and w'_* as a consequence of the discussion enclosed by the two agents, and also by the influence of external factors such as media or propaganda, and spontaneous changes of mind. Denoting by $f(t, w)$ the proportion of agents in the population with opinion

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w at time t, it is possible to describe the time evolution of $f(t,.)$ with a Boltzmann-like equation, whose collision part reflects the dynamics in the changes of opinion due to encounters. Thus, the long-time asymptotic behavior of $f(t,.)$ can be analyzed theoretically and/or numerically.

This procedure, which is by far not the only way of modelling opinion formation, is strongly inspired by the kinetic theory of rarefied gases and granular flows. The recent advances in the mathematical foundations of kinetic theory (see [49, 51]), motivated several mathematicians to perform a rigorous study of this kind of problems, using tools from partial differential equations, optimal transport, game theory and stochastic processes. This approach has been successfully implemented by Bellomo, Ben-Naim, Pareschi, Toscani and their collaborators in a wide variety of settings, we refer the interested reader to their works [2, 4, 5, 6, 7, 41] and the surveys in Ref. [40] for further details.

In this work we introduce a continuous model of opinion formation, where agents opinions are real numbers in $[-1, 1]$, and agents change their opinions through binary interactions as mentioned before. Most of the agents are assumed to have some propensity to reach an agreement, the so-called compromise hypothesis and hence after each interaction they tend to get closer positions. However, we introduce a high degree of heterogeneity among the agents, and each agent i has a priori two individual characteristics:

- some power of persuasion, represented by a probability $p_i \in [0, 1]$ that the agent will convince the other agent involved in the interaction, and
- some willingness to change his/her own opinion, represented by a probability $q_i \in [0, 1]$ that the agent is persuaded.

Observe that the assumption that q could be zero introduces *zealots* or stubborn individuals, i.e., agents who have strong opinions and they are not affected by other agents' opinions, not changing their mind after interactions. The presence of stubborn agents was studied mainly in discrete models of opinion dynamics, related to consensus formation, game theory models, and diffusion of innovations, among other applications, see [18, 36, 39, 54, 57, 56, 58].

In these works it is shown, mainly through simulations, how the stubborn agents affect the process of consensus formation, specially the kind of expected equilibria that could arise due to their influence, and the time to convergence. Let us remark that in [58] and related works, several results were proved theoretically using probabilistic arguments. Also, a striking fact was observed in the simulations: the time to convergence decreases when the number of stubborn agents increases.

Much fewer in number are the works considering continuous opinion models with zealots or persuasion, see for instance [11, 30]. Let us note that the presence of leaders and followers as in During et al. [22] has a somewhat similar dynamics when a leader and a follower interact, since only the follower updates its opinion. However, the interactions among leaders are allowed, and hence they can change their opinions.

In the work [23], the authors presented a related model including an additional variable representing the assertiveness level of agents, similar to our variable p. Now, the assertiveness evolves in time, and a Matthew effect, or Rich-Gets-Richer dynamics, is proposed, where after collisions the agent with higher (respectively, lower) assertiveness increases (resp., decreases) its value. On the other hand, the agents in their model are always receptive to other agents opinions, and no zealots are present.

Recently, in [55], each agent has some parameter k, which is a mix between zealotry and assertiveness, which also evolve in time, and zealots can appear dynamically in the model.

Our aim in this work is to rigorously show that the long time behavior of the agent based model can be described with a Boltzmann-like equation satisfied by the distribution of agents $f(t,.)$, and it is properly approximated by a non-linear non-local transport equation, which is well-posed for measure-valued functions. We then establish the convergence of the solution to some limit density, with explicit bounds on the time of convergence.

Essentially, the limit reveals that the part of the population composed by individuals who are willing to change their opinion tends to share the same opinion. Furthermore, we find out that this limit opinion is precisely the mean opinion of the stubborn individuals, those who always keep their own opinion ($q = 0$), weighted by their power of persuasion.

Moreover, the bounds for the rates of convergence point out that, the greater the number of stubborn individuals is, the faster the system reaches the stationary state. This fact has been observed in our simulations and also in related discrete opinion models, see for example [36, 38, 39, 56], exhibiting that the asymptotic distribution of opinions in the population is completely determined by the stubborn individuals, although their influence take a long time to be observed when there are just a few of them.

1.1. **Notations and definitions.** We denote $K = [-1,1] \times [0,1] \times [0,1]$ and a generic point of K as $\varpi = (w,p,q)$. Let $P(K)$ be the convex set of probability measures on K. Given $f \in P(K)$, we write the integral of a function ϕ against f as $\int_K \phi(\varpi)\, df(\varpi)$ or as $\int_K \phi(\varpi) f(\varpi)\, d\varpi$. The expression $f(\varpi)\, d\varpi$ is merely a notation, we are not assuming that f has a density necessarily. By $f(w)dw$ we understand the marginal of f with respect to the first variable w, namely

$$\int_{A\times[0.1]\times[0.1]} f(\varpi)\, d\varpi = \int_A f(w)dw \qquad \text{for any } A \subset [-1,1] \text{ Borel.}$$

Here again $f(w)dw$ is merely a notation, we are not assuming in general that this measure has a density.

$M(K)$ stands for the space of finite measures on K, $M_+(K)$ denotes the cone of nonnegative measures and $P(K)$ the convex cone of probability measures. These sets are endowed with the total variation norm, namely

$$\|f\| = \sup\left\{ \int_K \phi\, df : \phi \in C(K) \text{ such that } \|\phi\|_\infty \le 1 \right\}. \tag{1}$$

Let us remark that $M(K)$ becomes a Banach space with this norm.

Later on, we will need to endow the set $P(K)$ with the weak convergence of the measure topology. It will be convenient then, to recall from [50] that the Wasserstein distances W_p, with $p \geq 1$, between two probability measures μ, ν are defined as

$$W_p(\mu,\nu) = \left(\inf_{\alpha \in \Gamma(\mu,\nu)} \int_{K\times K} |x-y|^p d\alpha(x,y) \right)^{1/p}, \quad p \geq 1, \tag{2}$$

being $\Gamma(\mu,\nu)$ the collection of all measures on $K \times K$ with marginal measures ν and μ on the first and second factor, respectively. When $p=1$ the Kantorovich and Rubinstein Theorem provides a dual representation of W_1, namely

$$W_1(\nu,\mu) = \sup\left\{ \int_K \varphi \, d(\mu-\nu) : \quad \varphi \text{ is 1- Lipschitz} \right\}. \tag{3}$$

1.2. **Organization of the paper.** The paper is organized as follows.

Section §2 contains a detailed description of the rules governing the updates of the individuals opinions during encounters, determining a Boltzmann-like equation satisfied by the agent distribution $f(t,.)$. We introduce in addition the so-called grazing limit that yields a Fokker-Planck equation, modelling the long-time asymptotic behavior of the density, when the interactions among the agents produce very tiny changes in their opinions, namely when the parameters $\sigma, \gamma \to 0$. This idea of studying Boltzmann-like equation in the limit of small changes in each interaction comes from the literature about the Boltzmann equation (see e.g. [18, 19, 20, 52, 53] and references therein) and was first applied in the context of opinion formation model by Ben-Naim, Krapivsky and Redner [6], and by Toscani [47].

In Section §3 we derive to the analysis of the asymptotic behavior as $t \to +\infty$ of the equation arising when in the grazing limit $\frac{\sigma^2}{\gamma} \to 0$, namely, when the transport term dominates the diffusive term,

$$\partial_t f(w,p,q) + \partial_w((m_t - w)q\langle p\rangle f(w,p,q)) = 0,$$

where $\langle p\rangle$ is the mean value of the persuasion power p, which remains constant in time, and $m(t) = \int_K \frac{p}{\langle p\rangle} w \, df_t(\varpi)$ is the weighted mean opinion.

We characterize the limit distribution of agents, which consists of the original distribution of stubborn agents, plus a delta function at the mean of their opinions, weighted by they power of persuasion. We determine explicit bounds on the rate of convergence and show that the time to convergence decreases as the number of stubborn agents increases.

Finally, some computational experiments illustrating our theoretical results are included in Section §4. We perform an agent based simulation of the dynamics of our problem without noise, in order to verify that its stationary distribution coincide with the theoretical one.

In a Supplement we include for a sake of completeness the proof of the existence and uniqueness of solutions to the Boltzmann equation introduced in section 2, using the ideas in Chapter 6 of the book of Cercignani, Illner and Pulvirenti [15]. For the reader's convenience, also we provide in this Supplement a detailed proof of the approximation of the Boltzmann-like equation by a diffusion-transport equation, via the grazing limit. This proof is based mainly on Toscani [47].

2. Description of the model

2.1. **Microscopic interaction rules.** Let us introduce our model of opinion formation. We consider a population composed by N agents. The opinion of an agent with respect to certain matter is represented by a real number $w \in [-1,1]$ (meaning -1 being completely in disagreement with the subject and 1 in complete agreement). In addition, we take into account the ability (or difficulty) of an individual to persuade another agent, as well as his/her reticence (or facility) to change his/her opinion. We denote by $p \in [0,1]$ the probability of the agent to convince the opponent and by $q \in [0,1]$ the probability that the agent is persuaded to change his/her own opinion. Each agent is thus characterized by three parameters (w,p,q).

Agents' parameters (w,p,q) could be modified during binary encounters. For simplicity, in this work the parameters $(p;q)$ are assumed to be fixed and to remain unchanged in time, although there exist models where the agents' persuasion also evolve, as in [11, 48].

We now describe the up-dating rules of the opinions. Consider two interacting agents with parameters (w,p,q) and (w_*,p_*,q_*) before the encounter. Denote by (w',p',q') and (w'_*,p'_*,q'_*) the new values for the parameters after the interaction, respectively. As we mentioned before, the parameters (p,q) will remain unchanged: $p'=p$, $q'=q$, $p'_*=p_*$, $q'_*=q_*$. Regarding the up-dating of the opinion, we propose the following rule:

$$
\begin{aligned}
w' &= w + \gamma q p_*(w_* - w) + \eta q D(|w|),\\
w'_* &= w_* + \gamma p q_*(w - w_*) + \eta_* q_* D(|w_*|).
\end{aligned}
\tag{4}
$$

Observe that the change of opinion $w'-w$ is the sum of two parts. On the one hand, the term $\gamma q p_*(w_*-w)$ reflects the idea that the agents tend to reach a compromise. This tendency is directly proportional to both his/her willingness of changing his/her own opinion, q and also the power of persuasion of the opponent, p_*. Here γ is a given real number in $(0,1/2)$ modelling the strength of the interaction.

On the other hand, the term $\eta q D(|w|)$ represents the inclination of an agent to change his/her opinion due to random external or internal factors. Obviously, this term is proportional to the facility q of the agent to modify his/her opinion. By η and η_* we denote two independent and identically distributed random variables, with null expected value and variance σ^2. More precisely, we will write $\eta = \sigma Y$, being Y a symmetric random variable such that $E[Y]=0$, $Var[Y]=1$ and $E[|Y|^3]<\infty$, and the same is assumed for η_*. The function $D(|w|) \in [0,1]$ is supposed to be non-increasing in $|w|$. Some typical examples are $1-w^2$, $1-|w|$ and $\sqrt{1-w^2}$. Notice that in these examples $D(\pm 1)=0$. This is in accordance with the fact that the more extreme an opinion is, the more difficult to be changed.

2.2. **Macroscopic kinetic model: Boltzman equation.** Let $f(t,\varpi)$ be the distribution agents with opinion ϖ at time $t \geq 0$, hence $f(t,\cdot)$ is a probability measure on K. We usually denote this measure as df_t or $f_t(\varpi)d\varpi$ bearing in mind that f_t may not necessarily be absolutely continuous with respect to Lebesgue measure. In fact, f_t could be a Dirac measure.

In case of binary interactions the time evolution of the density f is a balance between gain and loss of opinion terms through an integro-differential equation of Boltzmann type:

$$\begin{aligned}\frac{d}{dt}\int_K \phi(\varpi)\,df_t(\varpi) \\ = \int_{B^2}\int_{K^2} \beta_{(w,w_*)\to(w',w'_*)}(\phi(\varpi')-\phi(\varpi))\,df_t(\varpi)df_t(\varpi_*)d\eta d\eta_*,\end{aligned} \tag{5}$$

for any $\phi \in C^\infty(K)$, see [15]. The kernel β is related to the transition rate and takes into account the external events acting on the opinion. For simplicity, we can take

$$\beta_{(w,w_*)\to(w',w'_*)} = \theta(\eta)\theta(\eta_*)\chi_{|w'|\le 1}\chi_{|w'_*|\le 1},$$

where by χ_A we understand the indicator function of the set A and θ is a symmetric probability density with zero mean and variance σ^2, characterizing the diffusion of information. To avoid the dependence of β on the probabilities w, w_* through the indicator function, we can ensure the boundedness of $|w'|$ and $|w'_*|$, assuming that the support of the random variables η, η_* is conveniently delimited. This reckons on the choice of the function D; for instance, if $D(|w|) = 1-|w|$ it suffices to take $B = (-(1-\gamma), 1-\gamma)$ to obtain $|w'| \le 1, \quad |w_*| \le 1$, while if $D(|w|) = 1-w^2$, it is enough to have $|\eta| \le \frac{1-\gamma}{2}$ since then $|\eta| \le \frac{1-\gamma}{1+|w|}$ (see [47]). With these choices, equation (5) corresponds to a classical Boltzmann equation

$$\frac{d}{dt}\int_K \phi(\varpi)\,df_t(\varpi) = \int_B\int_{K^2}(\phi(\varpi')-\phi(\varpi))\,df_t(\varpi)df_t(\varpi_*)d\theta(\eta), \tag{6}$$

for any $\phi \in C^\infty(K)$.

Taking $\phi(\varpi) = p$ and $\phi(\varpi) = q$ as test functions in (6) it is easy to see that the average of the persuasion ability, $\langle p\rangle$, and of the zealotry, $\langle q\rangle$, respectively, are constant in time. We assume that $\langle q\rangle > 0$, otherwise no opinion will change.

Our first concern is to show the existence of a solution to (6). This is the purpose of the following Theorem. The proof follows classical ideas and is detailed thereafter in the Supplement for the reader's convenience.

Thm 2.1. *Given $f_0 \in P(K)$, there exists a unique $f \in C^1([0,+\infty), P(K))$, where $P(K)$ is endowed with the total variation norm* (1)*, such that*

$$\begin{aligned}\int_K \phi(\varpi)df_t(\varpi) \\ = \int_K \phi(\varpi)df_0(\varpi) + \int_0^t\int_{K^2\times B}(\phi(\varpi')-\phi(\varpi))\,df_s(\varpi)df_s(\varpi_*)d\theta(\eta)ds,\end{aligned} \tag{7}$$

for any $\phi \in C(K)$.

2.3. **Grazing Limit.** Given some initial condition $f_0 \in P(K)$, consider the function f solution to the Boltzmann-like equation (6) given by Theorem 2.1. We will prove that, after an appropriate time rescaling, the asymptotic behavior of $f(t)$ as $t \to +\infty$ is well-described when $\gamma, \sigma \to 0$ by the solution $g \in C([0, +\infty), P(K))$ of some diffusion equation, whose form depends on the limit of the quotient $\frac{\sigma^2}{\gamma}$. Namely, it reckons on the balance between the diffusion strength, represented by σ and the tendency to an agreement, measured by the parameter γ. Indeed, we will see that in case they are proportional, i.e, $\sigma^2 = \gamma\lambda$ for some $\lambda > 0$, then g satisfies

$$\begin{aligned} \frac{d}{d\tau}\int_K \phi(\varpi) g_\tau(\varpi) = \quad & \int_K \Big((m(\tau) - w)\langle p\rangle q\Big)\partial_w\phi(\varpi)\, dg_\tau(\varpi) \\ & + \tfrac{\lambda}{2}\int_K q^2 D^2(|w|)\partial_{ww}\phi(\varpi)\, dg_\tau(\varpi), \end{aligned} \tag{8}$$

for any $\phi \in C^\infty(K)$. Here $m(\tau) = \int_K \frac{p}{\langle p\rangle} w\, dg_\tau(\varpi)$, where $\langle p\rangle$ is the mean value of p, which remains constant in time. In other words, m is the mean opinion weighted by the normalized power of persuasion.

Notice that (8) is the weak form of the Fokker-Planck equation

$$\partial_\tau g + \partial_w\Big((m(t) - w)q\langle p\rangle g\Big) = \frac{\lambda}{2}\partial_{ww}\Big(q^2 D(|w|)^2 g\Big), \tag{9}$$

subject to the following boundary conditions satisfied for any $\tau > 0$:

$$(m(\tau) - w)\langle p\rangle q g_\tau(\varpi) - \frac{\lambda}{2}\partial_w\Big(q^2 D^2(|w|) g_\tau(\varpi)\Big) = 0, \qquad w = \pm 1, (p, q) \in [0, 1]^2 \tag{10}$$

$$D(|w|)^2 \int_0^1 \int_0^1 q^2 g(\tau, w)\, dp dq = 0, \qquad w = \pm 1. \tag{11}$$

These conditions are the result of integrating by parts assuming that g_τ is smooth. In a wide choice of noise terms $D(\pm 1) = 0$ (e.g. if $D(|w|) = 1 - w^2$ or $D(|w|) = 1 - |w|$), thus (11) holds straightforward and (10) simplifies into

$$(m(\tau) - w)\langle p\rangle q g_\tau(\varpi) = 0, \qquad w = \pm 1, p \in [0, 1], q \in [0, 1]. \tag{12}$$

The left-hand side of (9) corresponds to a transport equation describing the tendency to agreement in the interacting rules. It amounts to a transport towards the mean opinion m with a velocity being proportional to $q\langle p\rangle$, the product between the tendency of an agent to change his opinion and the mean power of persuasion. The right-hand side of (9) is a diffusion term representing the possibility for an agent of changing his opinion under the influence of random external factors.

Notice that the limit equation (9) has both a diffusion and a transport term according to the assumption $\frac{\sigma^2}{\gamma} \to \lambda > 0$. If we suppose instead that $\frac{\sigma^2}{\gamma} \to 0$ or that $\frac{\sigma^2}{\gamma} \to +\infty$, the limit equation has only the transport term or the diffusion term, respectively.

Namely, if $\frac{\sigma^2}{\gamma} \to 0$, the limit equation turns out to be

$$\int_K \phi\, dg_\tau = \int_K \phi\, df_0 + \int_0^\tau \int_K (m(\tau) - w)\langle p\rangle q \phi_w(\varpi)\, dg_s(\varpi) ds,$$

which is the weak formulation to the transport equation

$$\partial_t f(w,p,q) + \partial_w((m_t - w)q\langle p\rangle f(w,p,q)) = 0.$$

We are interested here in this case, and the above mentioned facts regarding the grazing limit are summarized in the Supplement. We provide a full detailed proof based on the arguments in [47].

3. Asymptotic behavior of the Fokker-Planck equation without noise

This section is concerned with the asymptotic behavior as $t \to +\infty$ of solutions to the Fokker-Planck equation

$$\partial_t f(w,p,q) + \partial_w((m_t - w)q\langle p\rangle f(w,p,q)) = 0, \tag{13}$$

or its weak form (67). This equation arises when in the grazing limit dominates the transport term, namely $\frac{\sigma^2}{\gamma} \to 0$, see Theorem B.1.

Recall that $\langle p\rangle$ is the mean value of the persuasion power p, which remains constant in time and that $m(t) = \int_K \frac{p}{\langle p\rangle} w \, df_t(\varpi)$ is the weighted mean opinion.

The following observation ensures the uniqueness of solutions to (67).

Remark 3.1. *Given $f_0 \in P(K)$ and $f \in C([0,+\infty), P(K))$, $f(0) = f_0$, it is easily seen that the vector-field*

$$E(t,\varpi) := v[f_t](\varpi) := \Big(\Big[\int \frac{pw}{\langle p\rangle}\, df_t(\varpi) - w\Big] q\langle p\rangle, 0, 0\Big) = ((m(t) - w)q\langle p\rangle, 0, 0),$$

where $\langle p\rangle = \int_K p \, df_0(\varpi)$, satisfies the following:

(1) E is continuous in (t,ϖ),
(2) $|E(t,\varpi)| \le C$ for any (t,ϖ),
(3) $|E(t,\varpi) - E(t,\varpi')| \le C|\varpi - \varpi'|$ for any t, ϖ, ϖ'.

Moreover if $g \in C([0,+\infty), P(K))$, $g(0) = f_0$, then

$$\max_{\varpi \in K} |v[f_t](\varpi) - v[g_t](\varpi)| \le CW_1(f_t, g_t),$$

for any $t \ge 0$.

Invoking the theory developed in [14] *by Cañizo, Carrillo and Rosado, we can ensure that the equation $\partial_t f + div(v[f_t](\varpi) f_t) = 0$, which is exactly* (13)*, has a unique solution in $C([0,+\infty), P(K))$ with initial condition f_0.*

The long time behaviour of the solution will be accomplished by rewriting equation (13) in a simpler form due to Li and Toscani [35]. To apply this idea we need to bear in mind some facts about the generalized inverse of the cumulative distribution function of a probability measure. Only measures supported in $[-1,1]$ will be considered, since this is the case of interest in this paper, see the next subsection.

3.1. **A change of variable.** Let $f \in P([-1,1])$. The cumulative distribution function (cdf) $F : \mathbb{R} \to [0,1]$ of f is defined as $F(x) = f((-\infty, x])$. Notice that F is non-decreasing and right-continuous with left limit.

The generalized inverse of F is defined as $F^{-1} : [0,1] \to [-1,1]$

$$F^{-1}(\rho) = \inf\{x \in [-1,1] \text{ s.t. } F(x) \geq \rho\}. \tag{14}$$

Observe that F^{-1} is non-decreasing, left-continuous with right limit in $(0,1]$ and

$$[F^{-1}(0^+), F^{-1}(1)] \supset \operatorname{supp} f. \tag{15}$$

Furthermore, for any $x \in [-1,1]$ and any $\rho \in [0,1]$ the following inequalities hold:

$$\text{If } F(x) > 0 \text{ then } F^{-1}(F(x)) \leq x \text{ while } F(F^{-1}(\rho)) \geq \rho. \tag{16}$$

See the note of Embrechts and Hofert [24] for the above (and further) properties of F^{-1}.

The use of the generalized inverse enables us to rewrite an equation like (13) in terms of the generalized inverse of the cdf of f_t, and the resulting equation is usually much simpler. More precisely, consider $f \in C([0,\infty); P([-1,1]))$ and let F_t be the cdf of f_t and $X_t = F_t^{-1}$ its generalized inverse. Then, it can be proved that

$$\int_0^1 \phi(X_t(r))\, dr = \int_{-1}^1 \phi(w)\, df_t(w), \tag{17}$$

for any ϕ integrable (to prove this identity it suffices to check the formula for ϕ of the form $1_{(-\infty,a]}$, $a \in \mathbb{R}$). This change of variables formula is the key of the next result.

Proposition 3.1. *Let $v : [0,+\infty) \times [-1,1] \to \mathbb{R}$ be continuous and globally Lipschitz with respect to the second variable. Then $f \in C([0,+\infty), P([-1,1]))$ is a weak solution of*

$$\partial_t f_t + \partial_x (v(t,x) f_t) = 0, \tag{18}$$

in the sense that for any $\phi \in C^\infty([-1,1])$ and any $t > 0$,

$$\int_{-1}^1 \phi(x)\, df_t(x) = \int_{-1}^1 \phi(x)\, df_0(x) + \int_0^t \int_{-1}^1 \phi'(x) v(s,x)\, df_s(x) ds, \tag{19}$$

if and only if for any $r \in (0,1]$, $X_t(r)$ is a solution of

$$\partial_t X_t(r) = v(t, X_t(r)). \tag{20}$$

Here X_0 is the generalized inverse of F_0 (the cdf of f_0).

The proof can be found essentially in Theorem 3.1 of Ref. [1]. However, we rewrite it here under the point of view of the ordinary equation for the flux (20).

Proof. Assume that X_t satisfies (20). Thanks to (17), for any smooth ϕ we have that

$$\begin{aligned} \frac{d}{dt} \int_{-1}^1 \phi(x)\, df_t(x) &= \frac{d}{dt} \int_0^1 \phi(X_t(r))\, dr = \int_0^1 \phi'(X_t(r)) v(t, X_t(r))\, dr \\ &= \int_{-1}^1 \phi'(x) v(t,x)\, df_t(x), \end{aligned}$$

which easily implies (19).

Reciprocally, suppose that f solves (18). By Fubini's theorem,

$$\begin{aligned}\int_{-1}^{1} \phi(x) F_t(x) dx &= \int_{-1}^{1} \phi(x) \int 1_{(-\infty,x]}(y)\, df_t(y) dx \\ &= \int_{-1}^{1} \Big(\int_{y}^{+\infty} \phi(x)\, dx \Big)\, df_t(y).\end{aligned}$$

Differentiating with respect to time and taking into account (18) yield

$$\frac{d}{dt} \int_{-1}^{1} \phi(x) F_t(x) dx = - \int_{-1}^{1} \phi(y) v(t,y)\, df_t(y).$$

Moreover, $\partial_x F_t = f_t$ in the distributional sense. Thus F is a weak solution of the transport equation

$$\partial_t F_t + v(t,x) \partial_x F_t = 0. \tag{21}$$

Let $\phi_t(x)$ be the flow of v, i.e. the solution to $\partial_t \phi_t(x) = v(t, \phi_t(x))$, starting at $\phi_0(x) = x$. Then, as usual F is determined by the relation $F_t(\phi_t(x)) = F_0(x)$. It is now simple to conclude that (20) holds i.e. that $X_t(r) = \phi_t(X_0(r))$.

First $F_t(\phi_t(X_0(r))) = F_0(X_0(r))$ which is greater than r by (16). Hence, for any t, $\phi_t(X_0(r)) \geq X_t(r)$.

Conversely, for any $x < X_0(r)$ we have $F_0(x) < r$ so that $F_t(\phi_t(x)) < r$ and then $\phi_t(x) < X_t(r)$ by definition of $X_t(r)$. Letting $x \to X_0(r)$ we obtain $\phi_t(X_0(r)) \leq X_t(r)$.

The proof is finished. □

3.2. **Conditional distributions.** Another useful tool to achieve the asymptotic analysis is the concept of conditional distribution.

Let X, Y be two random variables defined over the same probability space with values in $\mathbb{R}^d$ and $\mathbb{R}^k$, respectively, and denote by P_X and $P_{(X,Y)}$ the distributions of X and (X,Y). Then there exists a map $\nu : (x, B) \in \mathbb{R}^d \times \mathcal{F}(\mathbb{R}^k) \to \nu(x,B) \in [0,1]$ (where $\mathcal{F}(\mathbb{R}^k)$ is the Borel σ-field) such that:

i) $\nu(x,.) \in P(\mathbb{R}^k)$ for any $x \in \mathbb{R}^d$,
ii) $\nu(., B)$ is measurable for any $B \subset \mathbb{R}^k$ Borel,
iii) $P(X \in A; Y \in B) = \int_A \nu(x,B)\, dP_X(x)$ for any $A \subset \mathbb{R}^d$, $B \subset \mathbb{R}^k$ Borel.

We write $\nu(x,B) = P(Y \in B | X = x)$.

The following Fubini formula holds: for any $\phi : X \times Y \to \mathbb{R}$ $P_{(X,Y)}$-integrable, the function $x \to \int_Y \phi(x,y)\, P(dy|X = x)$ is measurable and

$$\int_{X\times Y} \phi\, dP_{(X,Y)} = \int_X \Big(\int_Y \phi(x,y)\, P(dy|X=x) \Big)\, dP_X(x). \tag{22}$$

Of course the same results can be written in terms of a probability measure $\mu \in P(\mathbb{R}^d \times \mathbb{R}^k)$ and its marginal in $\mathbb{R}^d$, μ_1. In that case we let $\mu_{|x} := \nu(x,.)$ and any μ-integrable ϕ satisfies

$$\int_{X\times Y} \phi\, d\mu = \int_X \Big(\int_Y \phi(x,y)\, \mu_{|x}(y) \Big)\, d\mu_1(x). \tag{23}$$

The existence of ν is guaranteed by Jirina's theorem. There are several classical references in this subject, see for details [3, 12, 33, 46].

3.3. **The asymptotic behavior of solutions.** We are now ready to analyze how the interaction of stubborn agents with those more likely to change their opinions affects the population's opinion dynamics. Indeed, the agents with fixed opinion will drag the opinion of the rest of the individuals to certain average of their own initial distribution, no matter the initial distribution considered for the whole population, see Theorems 3.1 and 3.2 below.

On the contrary, the asymptotic behavior when $q > 0$ for every agent, which will be studied in a future work, takes into account the values of the initial distribution of every individual.

Precisely, we consider an initial distribution $f_0 \in P(K)$ of the form

$$f_0(w,p,q)dwdpdq = \alpha_0 f_0^0(w,p)dwdp \otimes \delta_{q=0} + (1-\alpha_0)f_0^1, \tag{24}$$

for some $\alpha_0 \in (0,1]$, where $f_0^1 \in P(K)$ is supported in $\{q \geq \varepsilon\}$ for some $\varepsilon > 0$, and f_0^0 is a probability measure on $[-1,1] \times [0,1]$. This means that there exists a positive fraction α_0 of stubborn people whose opinion is distributed according to f_0^0, and that the parameters (w,p,q) of the rest of the population verify $q \geq \varepsilon$ and are determined by f_0^1.

Notice that the dynamics described in (4) deny changes in (p,q) for each agent and in consequence, the solution f_t of (67) with initial condition f_0 given in (24) will have the form

$$f_t(w,p,q)dwdpdq = \alpha_0 f_0^0(w,p)dwdp \otimes \delta_{q=0} + (1-\alpha_0)f_t^1(w,p,q). \tag{25}$$

We prove that in this case the non-stubborn agents share asymptotically the same opinion m_∞, which is completely determined by the opinion of the stubborn individuals. Indeed, we shall see that m_∞ is the mean opinion of the stubborn people weighted by their persuasion power.

The occurrence of this fact is specially well observed if we assume that the marginal in (p,q) of the distribution of the opinion among the non-stubborn population, $f_0^1(p,q)dpdq$, is given by a finite convex combination $\sum_{i=1}^N \alpha_i \delta_{p=p_i,q=q_i}$ of Dirac masses. As we will see, the analysis of the asymptotic behavior of the opinion distribution $f_t^i(w)dw$ of the population with $(p,q) = (p_i,q_i)$ can be conveniently reduced to the study of a linear system of ordinary equations $M' = AM + B$ in $\mathbb{R}^N$.

It is known (see [8]) that any probability measure $\mu \in P(\mathbb{R}^d)$ can be approximated with high probability by the empirical measure $\hat{\mu}^N := \frac{1}{N}\sum_{i=1}^N \alpha_i \delta_{X_i}$, being $X_1, .., X_N$ N random variables identically distributed with law μ. Then, it is reasonable to think that the results obtained for the discrete model enlighten the asymptotic behavior of the general case, as it indeed occurs.

Therefore, we first examine the simplified discrete system to provide us with some intuition, before accomplishing the proof for any general initial distribution given by (24). This is the core of the following theorem.

Thm 3.1. *Let $f_0 \in P(K)$ be an initial distribution defined as in* (24)*, where the initial distribution f_0^1 of the variables (w,p,q) corresponding to the non-stubborn population has the form*

$$f_0^1 = \sum_{i=1}^{N} \alpha_i g_0^i(w)dw \otimes \delta_{p=p_i, q=q_i}, \tag{26}$$

being $N \in \mathbb{N}$, $\alpha_1,..,\alpha_N > 0$ with $\alpha_1 + .. + \alpha_N = 1$, $q_1,..,q_N \in [\varepsilon, 1]$ all distinct, $p_1,..,p_N \in (0,1]$, and $g_0^1,..,g_0^N \in P([-1,1])$. Its evolution in time, f_t^1, verifies

$$W_1(f_t^1, \delta_{m_0^0} \otimes \sum_{i=1}^{N} \alpha_i \delta_{p=p_i,q=q_i}) \to 0, \qquad \text{as } t \to +\infty. \tag{27}$$

Here m_0^0 denotes the mean opinion of the stubborn people weighted by the power of persuasion, namely

$$m_0^0 = \frac{1}{\langle p\rangle_{q=0}} \int_{-1}^{1}\int_0^1 pw\, df_0^0(w,p). \tag{28}$$

Remark 3.2. *An estimation of the velocity of convergence in* (27) *will be determined for the general case in Theorem 3.2.*

Furthermore, for an intuitive explanation for the fact that the opinion of the non-stubborn agents converges to m_0^0, see Remark 3.4 below.

Proof. Observe that the distribution f_t^1 in (25) has the form:

$$f_t^1 = \sum_{i=1}^{N} \alpha_i g_t^i(w)dw \otimes \delta_{p=p_i, q=q_i}, \tag{29}$$

with $g_t^1,..,g_t^N \in P([-1,1])$. Notice that for any $i = 1,..,N$,

$$f^1_{t|p=p_i,q=q_i} = g_t^i$$

and

$$\partial_t g_t^i(w) + \partial_w((m_t - w)q_i\langle p\rangle g_t^i(w)) = 0, \tag{30}$$

in weak formulation, namely for any $\phi \in C^1([-1,1])$,

$$\frac{d}{dt}\int_{-1}^{1} \phi(w)\, dg_t^i(w) = \int_{-1}^{1} (m_t - w)q_i\langle p\rangle \phi'(w)\, dg_t^i(w). \tag{31}$$

This follows from (67) extending ϕ to a smooth function with support in $[-1,1] \times (p_i - \eta, p_i + \eta) \times (q_i - \eta, q_i + \eta)$ with $\eta > 0$ small enough so that $(p_i - \eta, p_i + \eta) \times (q_i - \eta, q_i + \eta)$ does not contain any other (p_j, q_j).

Let us study the behavior of g_t^i, $i = 1,..,N$, as $t \to +\infty$. We claim that for any $i = 1,..,N$ and $t > 0$,

$$W_1(g_t^i, \delta_{m_t^i}) \le 2e^{-\varepsilon\langle p\rangle t}, \tag{32}$$

where

$$m_t^i := \int_{-1}^{1} w\, dg_t^i(w)$$

is the mean opinion of agents with $(p, q) = (p_i, q_i)$. Indeed, according to Proposition 3.1, it follows from (30) that the generalized inverse X_t^i of the cumulative distribution function corresponding to g_t^i satisfies

$$\partial_t X_t^i(r) = (m_t - X_t^i(r)) q_i \langle p \rangle,$$

for any $r \in (0, 1]$. Then,

$$\partial_t (X_t^i(1) - X_t^i(r))^2 = -2 q_i \langle p \rangle (X_t^i(1) - X_t^i(r))^2,$$

so that by Gronwall's Lemma

$$X_t^i(1) - X_t^i(r) \le (X_0^i(1) - X_0^i(r)) e^{-q_i \langle p \rangle t} \le 2 e^{-\varepsilon \langle p \rangle t}. \tag{33}$$

On the other hand, since $m_t^i = \int_0^1 X_t^i(r)\, dr$, it holds that $X_t^i(0^+) \le m_t^i \le X_t^i(1)$ for all t. As a result,

$$\begin{aligned} W_1(g_t^i, \delta_{m_t^i}) &\le \int_{-1}^1 |w - m_t^i| \, dg_t^i(w) = \int_0^1 |X_t^i(r) - m_t^i| \, dr \\ &\le |X_t^i(1) - X_t^i(0^+)|, \end{aligned}$$

which, combined with (33), gives (32).

In view of (32), it is natural to study the asymptotic behavior of m_t^i, $i = 1, .., N$. Taking $\phi(w) = w$ in (31), we obtain

$$\frac{d}{dt} m_t^i = (m_t - m_t^i) q_i \langle p \rangle.$$

According to (25) and (29), we have

$$\begin{aligned} \langle p \rangle m_t &= \int_{-1}^1 \int_0^1 \int_0^1 p w \, df_t(w, p, q) \\ &= \alpha_0 \langle p \rangle_{|q=0} m_0^0 + (1 - \alpha_0) \sum_{j=1}^N \alpha_j p_j m_t^j, \end{aligned}$$

where m_0^0 is defined in (28). Thus for any $i = 1, .., N$,

$$\begin{aligned} \frac{1}{q_i} \frac{d}{dt} m_t^i = {} & \Big((1 - \alpha_0) \alpha_i p_i - \langle p \rangle \Big) m_t^i \\ & + (1 - \alpha_0) \sum_{k=1..N,\, k \neq i} \alpha_k p_k m_t^k + \alpha_0 \langle p \rangle_{|q=0} m_0^0. \end{aligned} \tag{34}$$

Introducing $M(t) := (m_t^1, .., m_t^N)^T$, this can be rewritten as

$$M'(t) = A M(t) + B,$$

with $B = \alpha_0 \langle p \rangle_{|q=0} m_0^0 (q_1, .., q_N)^T$ and $A = (a_{ij})_{ij}$ with

$$a_{ij} = \begin{cases} q_i \Big((1 - \alpha_0) \alpha_i p_i - \langle p \rangle \Big) & \text{if } j = i \\ (1 - \alpha_0) q_i \alpha_j p_j & \text{if } j \neq i. \end{cases}$$

The solution is explicitly

$$M(t) = e^{tA}M(0) + \int_0^t e^{(t-s)A}B\,ds = e^{tA}(M(0) + A^{-1}B) - A^{-1}B.$$

Notice that for any $i = 1, .., N$,

$$a_{ii} = -q_i\Big(\langle p\rangle_{|q=0}\alpha_0 + (1-\alpha_0)\sum_{j=1,..,N,\,j\neq i}\alpha_j p_j\Big).$$

It is then easily seen that $A(1, .., 1)^T = -\alpha_0\langle p\rangle_{|q=0}(q_1, .., q_N)^T$, which yields that

$$A^{-1}B = -m_0^0(1, \dots, 1)^T.$$

According to Gerschgorin's disc theorem,

$$\sigma(A) \subset \bigcup_{i=1}^N D\Big(a_{ii}, \sum_{k=1..N,k\neq i}|a_{ik}|\Big),$$

where $\sigma(A)$ denotes the spectrum of A and $D(z, r)$ the disc in the complex plane centered at z of radius r. Thus, for any $i = 1, .., N$,

$$a_{ii} + \sum_{k=1..N,k\neq i}|a_{ik}| = -q_i\langle p\rangle_{q=0}\alpha_0 \le -\varepsilon\alpha_0\langle p\rangle_{q=0}, \tag{35}$$

which implies that

$$\sigma(A) \subset \{z \in \mathbb{C} :\ Re(z) \le -\varepsilon\alpha_0\langle p\rangle_{q=0}\}. \tag{36}$$

We thus deduce that as $t \to +\infty$, $M(t) \to -A^{-1}B = m_0^0(1, .., 1)^T$ exponentially fast. This together with (32) is enough to obtain that $W_1(g_t^i, \delta_{m_0^0}) \to 0$ for any $i = 1, \dots, N$. We are ready now to show (27). Using that W_1 combines properly with convex combinations (see [50]), we have

$$\begin{aligned} W_1(f_t^1, \sum_{i=1}^N \alpha_i\delta_{m_0^0}\otimes\delta_{p=p_i,q=q_i}) &\le \sum_{i=1}^N \alpha_i W_1(g_t^i\otimes\delta_{p=p_i,q=q_i}, \delta_{m_0^0}\otimes\delta_{p=p_i,q=q_i}) \\ &\le \sum_{i=1}^N \alpha_i W_1(g_t^i, \delta_{m_0^0}), \end{aligned} \tag{37}$$

which goes to 0 as $t \to +\infty$. This completes the proof. □

Remark 3.3. *The problem without the presence of stubborn agents will be treated in a forthcoming work. Observe that in this case the inequality* (35) *is no longer strictly negative. Therefore, this situation requires very different arguments.*

We now study the general case:

Thm 3.2. *Assume that the initial distribution has the form*

$$f_0 = \alpha_0 f_0^0 + (1-\alpha_0)f_0^1,$$

where $f_0^0 \in P(K)$ is supported in $\{q = 0\}$ and $f_0^1 \in P(K)$ is supported in $\{q \ge \varepsilon_0\}$. Admit also that the map

$$(p, q) \in [0, 1] \times [0, \varepsilon_0] \to f^1_{0|(p,q)} \in P([-1, 1])$$

is globally Lipschitz for the W_1*-distance: there exists* $L > 0$ *such that for any* (p,q), $(p',q') \in [0,1] \times [\varepsilon_0, 1]$,

$$W_1(f^1_{0|(p,q)}, f^1_{0|(p',q')}) \leq L(|q-q'| + |p-p'|). \tag{38}$$

Then,

$$W_1(f^1_t, f^1_0(p,q)dpdq \otimes \delta_{m^0_0}) \leq 4e^{-\alpha_0\varepsilon_0\langle p\rangle_{|q=0} t}, \tag{39}$$

where

$$m^0_0 := \int \frac{p}{\langle p\rangle_{|q=0}} w\, df^0_0(w,p) \tag{40}$$

is the mean opinion weighted by the normalized persuasion power within the group of stubborn agents. Here $\langle p\rangle_{|q=0} = \int p\, df^0_0(p)$ *stands for the mean value of* p *among the stubborn agents.*

Remark 3.4. *Let us give an intuitive motivation for the convergence of the opinion of the non-stubborn agents to* m^0_0*. Suppose we know that there exists* $m_\infty := \lim_{t\to+\infty} m_t$*. In view of equation* (13)*, it seems reasonable to conjecture that* f^1_t *converges to* $\delta_{m_\infty}(w) \otimes f^1_0(p,q)dpdq$ *so that* $f_t \to \alpha_0 f^0_0 + (1-\alpha_0)\delta_{m_\infty}(w) \otimes f^1_0(p,q)dpdq$*. In particular we can pass to the limit in the definition of* m_t *to obtain that*

$$\langle p\rangle m_\infty = \langle p\rangle \lim_{t\to+\infty} m_t = \alpha_0 \int pw\, df^0_0 + (1-\alpha_0)m_\infty \int p\, df^1_0(p,q).$$

Moreover,

$$\langle p\rangle = \alpha_0 \int p\, df^0_0(p) + (1-\alpha_0)\int p\, df^1_0(p) = \alpha_0\langle p\rangle_{|q=0} + (1-\alpha_0)\int p\, df^1_0(p).$$

Furthermore,

$$\alpha_0\langle p\rangle_{|q=0} m_\infty = \alpha_0 \int pw\, df^0_0,$$

which implies that $m_\infty = m^0_0$*.*

Remark 3.5. *Sufficient conditions on* f^1_0 *ensuring the regularity assumption* (38) *can be easily found. Suppose for instance that* f^1_0 *has a density in the sense that* $f^1_0 = f^1_0(w,p,q)dwdpdq$ *with* $f^1_0 \in L^1(K)$*. Then,* $f^1_{0|(p,q)}(w) = \frac{f^1_0(w,p,q)}{f^1_0(p,q)}$ *if* $f^1_0(p,q) \neq 0$*. Let us assume that*

(1) $0 < C \leq f^1_0(p,q) \leq C' < \infty$ *for any* (p,q) *such that* $f^1_0(p,q) \neq 0$,
(2) there exists $C'' > 0$ *such that*

$$|f^1_0(w,p,q) - f^1_0(w,p',q')| \leq C''(|p-p'| + |q-q'|),$$

for any w *and any* $(p,q),(p',q')$ *with* $f^1_0(p,q), f^1_0(p',q') \neq 0$.

In that case $f^1_{0|(p,q)}$ *verifies*

$$|f^1_{0|(p,q)}(w) - f^1_{0|(p',q')}(w)| \leq C''(|p-p'| + |q-q'|),$$

for any w and any $(p,q),(p',q')$ with $f_0^1(p,q), f_0^1(p',q') \neq 0$. Consequently, for any $\phi : [-1,1] \to \mathbb{R}$ 1-Lipschitz and any $(p,q),(p',q')$ such that $f_0^1(p,q), f_0^1(p',q') \neq 0$, there holds

$$\int_{-1}^{1} \phi(w)\,(df^1_{0|(p,q)}(w) - df^1_{0|(p',q')}(w)) = \int_{-1}^{1} \phi(w)\,(f^1_{0|(p,q)}(w) - f^1_{0|(p',q')}(w))dw$$
$$\leq C''(|p-p'| + |q-q'|) \int_{-1}^{1} |\phi(w)|\,dw.$$

Without loss of generality, it can be assumed that $\phi(-1) = 0$, since the above inequalities are still valid when adding a constant to ϕ. Accordingly, $\|\phi\|_\infty \leq 2$. Taking the supremum over such ϕ in the above expression gives (38) with $L = 4C''$.

In the course of the proof we will use the following envelope Theorem due to Milgrom and Segal in [37]:

Thm 3.3. *Let X be a set. Consider the function $V(t) := \max_{x\in X} h(x,t)$, $t \in [0,1]$. Admit that h is absolutely continuous with respect to t for any x and there exists $b \in L^1([0,1])$ such that $|\partial_t h(x,t)| \leq b(t)$ for any $x \in X$ and almost any $t \in [0,1]$. Then V is absolutely continuous.*

Assume in addition that h is differentiable in t for any $x \in X$ and that for any $t \in [0,1]$ the set $X(t) := \operatorname{argmax} h(.,t)$ is non-empty. In this case, for any selection of $x^*(t) \in X(t)$ we have

$$V(t) = V(0) + \int_0^t \partial_t h(x^*(s),s)\,ds.$$

3.4. **The proof of Theorem 3.2.** We have all of the ingredients to show the asymptotic behaviour in the general case. For convenience, we divide the proof in several steps.

Proof of Theorem 3.2. For any t and any $p,q \in [0,1]\times[\varepsilon_0,1]$ denote by $f^1_{t|(p,q)} \in P([-1,1])$ the conditional distribution of opinion among the agents with parameter (p,q).

Step 3.1. *For any $(p,q) \in supp\ (f_0(p,q)dpdq)$, $f^1_{t|(p,q)}$ is the unique solution to*

$$\left\{ \begin{array}{l} \partial_t f^1_{t|(p,q)} + \partial_w((m_t - w)q\langle p\rangle f^1_{t|(p,q)}) = 0, \\ f^1_{t=0|(p,q)} = f_{0|(p,q)}, \end{array} \right. \tag{41}$$

in $C([0,+\infty), P([-1,1]))$.

Moreover, the function $(p,q) \to f^1_{t|(p,q)}$ is Lipschitz with respect to the Wasserstein distance W_1. Namely, for any $(p,q),(p',q') \in [0,1]\times[\varepsilon_0,1]$,

$$W_1(f^1_{t|(p,q)}, f^1_{t|(p',q')}) \leq C_t(|q-q'| + |p-p'|). \tag{42}$$

Furthermore, it fulfils

$$\int_K \phi\, df_t^1 = \int_0^1\int_0^1 \Big(\int_{-1}^1 \phi\, df^1_{t|(p,q)}(w)\Big)\, df_0^1(p,q), \qquad \forall\, \phi \in C(K). \tag{43}$$

Proof. The existence of a unique solution to (41) is ensured by the results of Cañizo, Carrillo y Rosado [14], see Remark 3.1.

Denote by ϕ_t the flow of the vector-field $(w,p,q) \to (q\langle p\rangle(m_t - w), 0, 0)$. Since m_t is considered to be a known C^1 function, this flow can be rewritten as $\phi_t(w,p,q) = (\phi_t^1(w,p,q), p, q)$ and $f_t^1 = \phi_t \sharp f_0^1$ being the push-forward measure defined as

$$\int_K \psi(w,p,q)\, df_t^1(w,p,q) = \int_K \psi(\phi_t(w,p,q))\, df_0^1(w,p,q),$$

for all $\psi \in C(K)$.

This implies that for any $\psi \in C([-1,1])$ and $\phi \in C([0,1]\times[0,1])$,

$$\int_K \psi(w)\phi(p,q)\, df_t^1(w,p,q) = \int_K \psi(\phi_t^1(w,p,q))\phi(p,q)\, df_0^1(w,p,q),$$

i.e.

$$\begin{aligned}&\int_0^1\int_0^1 \phi(p,q)\Big(\int_{-1}^1 \psi(w)\, df^1_{t|(p,q)}(w)\Big)\, df_0^1(p,q)\\ &= \int_0^1\int_0^1 \phi(p,q)\Big(\int_{-1}^1 \psi(\phi_t^1(w,p,q))\, df^1_{0|(p,q)}(w)\Big)\, df_0^1(p,q).\end{aligned}$$

The arbitrariness of $\phi \in C([0,1]\times[0,1])$ yields, for any $t \geq 0$ and any continuous function ψ, that

$$\int_{-1}^1 \psi(w)\, df^1_{t|(p,q)}(w) = \int_{-1}^1 \psi(\phi_t^1(w,p,q))\, df^1_{0|(p,q)}(w), \tag{44}$$

for almost any (p,q), except for a $f_0^1(p,q)dpdq$-null set.

In particular, for any $k \in \mathbb{N}$, there exists a $f_0^1(p,q)dpdq$-null set, denoted as $A_{t,k} \subset [0,1]\times[0,1]$, for which $\psi(w) = w^k$ verifies the previous inequality at any $(p,q) \in A^c_{t,k}$.

Note that $A_t := \cup_{k\geq 0} A_{t,k}$ is a $f_0^1(p,q)dpdq$-null set such that (44) holds for any polynomial ψ and any $(p,q) \in A_t^c$. The density of the polynomials in $C([-1,1])$ implies that indeed, (44) holds for any $\psi \in C([-1,1])$ and any $(p,q) \in A_t^c$ with A_t of $f_0^1(p,q)dpdq$-null measure. This equality can then also be expressed as

$$f^1_{t|(p,q)} = \phi_t^1(.,p,q)\sharp f^1_{0|(p,q)}, \tag{45}$$

for any $(p,q) \in A_t^c$.

We would like this identity is fulfilled for any $(p,q) \in \mathrm{supp}(f_0^1(p,q)dpdq)$. So, let us fix certain $t \geq 0$ and assume for the moment that there exists a constant $C_t > 0$ depending only on t such that for any $(p,q),(p',q') \in [0,1]\times[\varepsilon_0,1]$,

$$W_1(\phi_t^1(.,p,q)\sharp f^1_{0|(p,q)}, \phi_t^1(.,p',q')\sharp f^1_{0|(p',q')}) \leq C_t(|q-q'| + |p-p'|). \tag{46}$$

Note that this claim also shows the Lipschitz continuity stated in (42).

Using the decomposition $f_0^1(p,q)dpdq = f_0^{1,\,non-atom} + f_0^{1,\,atom}$, observe that (45) is also satisfied for any (p,q) belonging to a larger set, $A_t^c \cup \{f_0^{1,\,atom} > 0\}$, since $f_0^{1,\,atom}$ gives positive mass to each of its atoms. This observation and the continuity given in (42) conclude that (45) is satisfied in $A_t^c \cup \mathrm{supp}(f^{1,\,atom})$.

It remains to modify the definition of $f^1_{t|(p,q)}$ at the variables $(p,q)\in A_t\cap\mathrm{supp}(f_0^{1,\,non-atom})$ in such a way that $f^1_{t|(p,q)}$ preserves its continuity in t and (45) holds for any $(p,q)\in\mathrm{supp}(f^1_0(p,q)dpdq)$.

Take first some $(p,q)\in A_t\cap\{f_0^{1,\,non-atom}>0\}$. Since A_t has null measure, there exists a sequence $(p_k,q_k)\in A_t^c\cap\{f_0^{1,\,non-atom}>0\}$ such that $(p_k,q_k)\to(p,q)$. As a consequence of (42), $(f_{t|(p_k,q_k)})_k$ is a Cauchy sequence in the complete space $(P([-1,1],W_1)$, hence it converges to some limit $g_{(p,q)}\in P([-1,1])$. Furthermore, (42) ensures also that this limit does not depend on the approximating sequence $(p_k,q_k)_k$.

We then declare $f^1_{t|(p,q)}:=g_{(p,q)}$ on $A_t\cap\{f_0^{1,\,non-atom}>0\}$. That way $f^1_{t|(p,q)}$ is continuous and (45) holds for any $(p,q)\in\{f^{1,\,non-atom}>0\}$. Proceed with $(p,q)\in$ supp $(f^{1,\,non-atom})$ similarly, taking an approximating sequence $(p_k,q_k)\in\{f^{1,\,non-atom}>0\}$.

In conclusion, redefining $f^1_{t|(p,q)}$ on $f^1_0(p,q)dpdq$-null sets in such a way that $f^1_{t|(p,q)}$ and the right hand side of (45) are continuous with respect to (p,q), guarantees that (45) and (44) are satisfied for any $t\ge 0$ and any $(p,q)\in$ supp $(f^1_0(p,q)dpdq)$. Moreover, it is clear that this modification is in accordance with the application of Fubini's Theorem, thus (43) holds.

To conclude the proof, it remains to prove the claim (46). Let $\psi:[-1,1]\to\mathbb{R}$ be 1-Lipschitz. Note that

$$
\begin{aligned}
&\int\psi\,d(\phi^1_t(.,p,q)\sharp f^1_{0|(p,q)}-\phi^1_t(.,p',q')\sharp f^1_{0|(p',q')})\\
&=\int\psi(\phi^1_t(w,p,q))\,df^1_{0|(p,q)}(w)-\int\psi(\phi^1_t(w,p',q'))\,df^1_{0|(p',q')}(w)\\
&=\int\psi(\phi^1_t(w,p,q))-\psi(\phi^1_t(w,p',q'))\,df^1_{0|(p,q)}(w)\\
&\quad+\int\psi(\phi^1_t(w,p',q'))\,d(f^1_{0|(p',q')}-f^1_{0|(p,q)})(w)=I+II.
\end{aligned}
\tag{47}
$$

The second term can be estimated using the definition of the Wasserstein distance W_1,

$$
\begin{aligned}
II\ &\le Lip(\psi(\phi^1_t(.,p',q')))W_1(f^1_{0|(p',q')},f^1_{0|(p,q)})\\
&\le Lip(\phi^1_t(.,p',q'))L(|q-q'|+|p-p'|),
\end{aligned}
$$

where L is the Lipschitz constant given by the assumption (38). On the other hand, the first term in (47) can be bounded as

$$
I\le\max_{|w|\le1}|\psi(\phi^1_t(.,p,q))-\psi(\phi^1_t(.,p',q'))|\le\max_{|w|\le1}|\phi^1_t(.,p,q)-\phi^1_t(.,p',q')|.
$$

Summarizing,

$$
\begin{aligned}
&\int\psi\,d(\phi^1_t(.,p,q)\sharp f^1_{0|(p,q)}-\phi^1_t(.,p',q')\sharp f^1_{0|(p',q')})\\
&\quad\le\max_{|w|\le1}|\phi^1_t(w,p,q)-\phi^1_t(w,p',q')|+Lip(\phi^1_t(.,p',q'))L(|q-q'|+|p-p'|).
\end{aligned}
$$

At this stage, recall that

$$\phi_t^1(w,p,q) = w + q\langle p\rangle \int_0^t (m_s - \phi_s^1(w,p,q))\, ds,$$

therefore,

$$\phi_t^1(w,p,q) - \phi_t^1(w,p',q') = (q-q') < p > \int_0^t (m_s - \phi_s^1(w,p',q'))\, ds$$
$$+q\langle p\rangle \int_0^t (\phi_s(w,p',q') - \phi_s(w,p,q))\, ds.$$

Taking now into account that $|\phi_t^1(w,p,q)| \leq 1$ for any (w,p,q), we infer that

$$|\phi_t(w,p,q) - \phi_t(w,p',q')| \leq 2t|q-q'| + \int_0^t |\phi_s(w,p',q') - \phi_s(w,p,q)|\, ds.$$

With the use of Gronwall's lemma this yields

$$|\phi_t^1(w,p,q) - \phi_t^1(w,p',q')| \leq 2te^t|q-\hat{q}|.$$

Similar arguments prove that for $w, \tilde{w} \in [-1,1$ and $p,q \in [0,1]$,

$$|\phi_t^1(w,p,q) - \phi_t^1(\tilde{w},p,q)| \leq e^t|w-\tilde{w}|,$$

hence $Lip(\phi_t^1(.,p,q)) \leq e^t$. In conclusion, we have shown that

$$\int \psi\, d(\phi_t^1(.,p,q)\sharp f^1_{0|(p,q)} - \phi_t^1(.,p',q')\sharp f^1_{0|(p',q')}) \leq C(t)(|q-q'| + |p-p'|).$$

The desired claim (46) follows now taking the supremum among all functions ψ 1-Lipschitz. □

Remark 3.6. *It would be natural to conjecture that the density $f^1_{t|(p,q)}$, modified in $f_0^1(p,q)dpdq$-null set as in Theorem 3.3, still defines a conditional density. Indeed, it is straightforward to see that*

$$\mu_w(p,q) := f^1_{t|(p,q)}(w) \in P([0,1]\times[0,1]) \text{ for any } w \in [-1,1],$$

and $P(X \in A; Y \in B) = \int_A \int_B d\mu_w(p,q)\, df^1_{t|(p,q)}(w)$ for any $A \subset [-1,1]$, $B \subset [0,1] \times [0,1]$ Borel sets. However, the fact that $f_t^1(.,B)$ is measurable for any $B \subset [0,1]\times[0,1]$ Borel, is not so immediate, and nevertheless is out of the scope of our results. In particular, for our proof it suffices with (43)*.*

We denote by

$$m(t,p,q) = \int_{-1}^1 w\, df^1_{t|(p,q)}(w), \tag{48}$$

the mean opinion among the agents with parameter $p,q \in [0,1]\times[\varepsilon_0,1]$.

Step 3.2. *There holds*

$$W_1(f^1_{t|(p,q)}, \delta_{m(t,p,q)}) \leq 2e^{-\varepsilon_0\langle p\rangle t}. \tag{49}$$

Proof. It can be deduced analogously to (32). □

In view of the previous step, it is natural to study the asymptotic behavior of the function $m(t,.)$ as $t \to +\infty$.

Step 3.3. *For any $t \geq 0$ and any $(p,q) \in [0,1] \times [\varepsilon_0, 1]$ the function $m(t,p,q)$ defined in (48) satisfies*

$$\begin{aligned} \partial_t m(t,p,q) &= q\alpha_0 \langle p \rangle_{|q=0} \Big[m_0^0 - m(t,p,q) \Big] \\ &\quad + (1-\alpha_0) q \int_K p' \Big[m(t,p',q') - m(t,p,q) \Big]\, df_0^1(w,p',q'). \end{aligned} \tag{50}$$

Proof. Given that $f^1_{t|(p,q)}$ fulfills (41) for any $(p,q) \in [0,1] \times [\varepsilon_0, 1]$, in particular

$$\begin{aligned} \partial_t m(t,p,q) &= \frac{d}{dt} \int_{-1}^{1} w\, df^1_{t|(p,q)}(w) = q\langle p \rangle \int_{-1}^{1} (m_t - w)\, df^1_{t|(p,q)}(w) \\ &= q\langle p \rangle (m_t - m(t,p,q)). \end{aligned} \tag{51}$$

Moreover according to the definition of m_t,

$$\begin{aligned} \langle p \rangle m_t &= \int_K pw\, df_t(w,p,q) \\ &= \alpha_0 \int_{-1}^{1} \int_0^1 pw\, df_0^0(w,p) + (1-\alpha_0) \int_K pw\, df_t^1(w,p,q), \end{aligned}$$

being

$$\begin{aligned} \int_K pw\, df_t^1(w,p,q) &= \int_0^1 \int_0^1 p \Big(\int_{-1}^1 w\, df^1_{t|(p,q)}(w) \Big)\, df_0^1(p,q) \\ &= \int_0^1 \int_0^1 pm(t,p,q)\, df_0^1(p,q). \end{aligned}$$

Denote as $\langle p \rangle_{|q=0} := \int p\, df_0^0(p,w)$, that is, the mean value of p among the agents with $q = 0$. We have

$$\langle p \rangle = \int_K p\, df_t(w,p,q) = \alpha_0 \langle p \rangle_{|q=0} + (1-\alpha_0) \int_K p\, df_t^1(w,p,q).$$

In terms of $\langle p \rangle$ and $\langle p \rangle m_t$ equation (51) is equivalent to

$$\begin{aligned} \frac{1}{q} \partial_t m(t,p,q) &= \alpha_0 \Big[\int_{-1}^{1} \int_0^1 pw\, df_0^0(w,p) - \langle p \rangle_{|q=0} m(t,p,q) \Big] \\ &\quad + (1-\alpha_0) \int_K p' \Big[m(t,p',q') - m(t,p,q) \Big]\, df_0^1(w,p',q'), \end{aligned}$$

which in view of the definition of m_0^0 in (40), can be rewritten as (50). □

At this stage, our aim is to determine the behavior as $t \to +\infty$ of the solution $m(t,p,q)$ to the linear system (50), which is exactly the system appearing in (34), when f_0^1 had the special form $f_0^1 = \sum_{i=1}^N \alpha_i g_0^i(w) dw \otimes \delta_{p=p_i, q=q_i}$.

The following step proves that $m(t,p,q)$ is Lipschitz continuous in (p,q) uniformly in t.

Step 3.4. *For any $\varepsilon_0 < \varepsilon < 2/(L\langle p\rangle)$ (where L is given in (38)), and for any $(p,q),(p',q') \in [0,1]\times[\varepsilon_0,1]$, there holds*

$$|m(t,p,q) - m(t,p',q')| \le \frac{2}{\varepsilon\langle p\rangle}(|q-q'| + |p-p'|). \tag{52}$$

Proof. Using (50) we have

$$\begin{aligned}
&\partial_t[m(t,p,q) - m(t,p',q')] \\
&= \alpha_0\langle p\rangle_{|q=0} m_0^0(q-q') - \alpha_0\langle p\rangle_{|q=0}(q-q')m(t,p,q) \\
&\quad -\alpha_0\langle p\rangle_{|q=0} q'[m(t,p,q) - m(t,p',q')] \\
&\quad +(1-\alpha_0)(q-q')\int_0^1\int_0^1 \tilde p[m(t,\tilde p,\tilde q) - m(t,p,q)]\,df_0^1(\tilde p,\tilde q) \\
&\quad -(1-\alpha_0)q'[m(t,p,q) - m(t,p',q')]\int_0^1\int_0^1 p\,df_0^1(p,q) \\
&= (q-q')\Big\{\alpha_0\langle p\rangle_{|q=0}[m_0^0 - m(t,p,q)] \\
&\qquad\qquad +(1-\alpha_0)\int_0^1\int_0^1 \tilde p[m(t,\tilde p,\tilde q) - m(t,p,q)]\,df_0^1(\tilde p,\tilde q)\Big\} \\
&\quad -[m(t,p,q) - m(t,p',q')]q'\langle p\rangle.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}|m(t,p,q) - m(t,p',q')|^2 \\
&= (q-q')\Big[m(t,p,q) - m(t,p',q')\Big]\Big\{\alpha_0\langle p\rangle_{|q=0}[m_0^0 - m(t,p,q)] \\
&\qquad\qquad +(1-\alpha_0)\int \tilde p[m(t,\tilde p,\tilde q) - m(t,p,q)]\,df_0^1(\tilde p,\tilde q)\Big\} \\
&\quad -|m(t,p,q) - m(t,p',q')|^2 q'\langle p\rangle.
\end{aligned}$$

Recalling that $|m_0^0|, |m(t,p,q)| \le 1$, it is straightforward to see that

$$\Big|\alpha_0\langle p\rangle_{|q=0}[m_0^0 - m(t,p,q)] + (1-\alpha_0)\int \tilde p[m(t,\tilde p,\tilde q) - m(t,p,q)]\,df_0^1(\tilde p,\tilde q)\Big| \le 2.$$

The fact that $q' \ge \varepsilon_0$ allows to deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}|m(t,p,q) - m(t,p',q')|^2 \\
&\le 2|q-q'|\Big|m(t,p,q) - m(t,p',q')\Big| - \varepsilon_0\langle p\rangle|m(t,p,q) - m(t,p',q')|^2 \\
&\le 2(|q-q'| + |p-p'|)\Big|m(t,p,q) - m(t,p',q')\Big| \\
&\quad -\varepsilon_0\langle p\rangle|m(t,p,q) - m(t,p',q')|^2.
\end{aligned}$$

Let $u(t) \ge 0$ be the solution to

$$\begin{cases} u'(t) &= 4(|q-q'| + |p-p'|)\sqrt{u(t)} - 2\varepsilon_0\langle p\rangle u(t), \\ u(0) &= |m(0,p,q) - m(0,p',q')|^2. \end{cases}$$

Note that $|m(t,p,q)-m(t,p',q')|^2 \leq u(t)$. Moreover, writing the equation for u as

$$u'(t) = 2\varepsilon_0\langle p\rangle\sqrt{u(t)}\Big(u^* - \sqrt{u(t)}\Big),$$

where

$$u^* := \frac{2(|q-q'|+|p-p'|)|}{\varepsilon_0\langle p\rangle},$$

we see that if $u(0) \leq (u^*)^2$, then $u(t) \leq (u^*)^2$ for any t and $u(t) \to (u^*)^2$. Observe that in view of the assumption (38),

$$\begin{aligned} u(0) &= |m(0,p,q)-m(0,p',q')|^2 = \Big|\int_{-1}^1 w\, df^1_{0|(p,q)}(w) - \int_{-1}^1 w\, df^1_{0|(p',q')}(w)\Big|^2 \\ &\leq \Big(W_1(f^1_{0|(p,q)}, f^1_{0|(p',q')})\Big)^2 \leq L^2(|q-q'|+|p-p'|)^2. \end{aligned}$$

Taking $\varepsilon > 0$ such that $\varepsilon_0 < \varepsilon < 2/(L\langle p\rangle)$, ensures that $u(0) \leq (u^*)^2$ and thus $u(t) \leq (u^*)^2$ for any t. It follows then $|m(t,p,q)-m(t,p',q')|^2 \leq u(t) \leq (u^*)^2$, which proves (52).

□

We have all of the ingredients to show the convergence of $m(t,p,q)$ to m_0^0:

Step 3.5. *For any $(p,q) \in supp\,(f_0^1(p,q)dpdq)$ and any $t \geq 0$ it holds that*

$$|m(t,p,q)-m_0^0| \leq \Big(\max_{(p,q)\in supp(f_0^1)} |m(0,p,q)-m_0^0|\Big)e^{-\varepsilon_0\alpha_0\langle p\rangle_{|t=0}t}.$$

Proof. Relation (50) implies that for any $q \in [\varepsilon_0, 1]$ and $t \geq 0$

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}|m(t,p,q)-m_0^0|^2 \\ &= \partial_t m(t,p,q)[m(t,p,q)-m_0^0] \\ &= -q\alpha_0\langle p\rangle_{|q=0}[m_0^0 - m(t,p,q)]^2 \\ &\quad + q(1-\alpha_0)[m(t,p,q)-m_0^0]\int_0^1\int_0^1 \tilde{p}\Big[m(t,\tilde{p},\tilde{q}) - m(t,p,q)\Big]\, df_0^1(\tilde{p},\tilde{q}). \end{aligned} \tag{53}$$

In particular, choosing $(p,q)=(p^*,q^*)$ a maximum point for $|m(t,.)-m_0^0|$ (its existence is ensured since $\mathrm{supp}(f_0^1(p,q)dpdq)$ is compact and $m(t,.)$ is continuous). Then,

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}|m(t,.)-m_0^0|^2_{|(p^*,q^*)}\\
&= -q^*\alpha_0\langle p\rangle_{|q=0}[m_0^0-m(t,p^*,q^*)]^2\\
&\quad +q^*(1-\alpha_0)[m(t,p^*,q^*)-m_0^0]\int_0^1\int_0^1 \tilde p\Big[(m(t,\tilde p,\tilde q)-m_0^0)+(m_0^0-m(t,p^*,q^*))\Big]\,df_0^1(\tilde p,\tilde q)\\
&= -q^*\alpha_0\langle p\rangle_{|q=0}[m_0^0-m(t,p^*,q^*)]^2\\
&\quad +q^*(1-\alpha_0)[m(t,p^*,q^*)-m_0^0]\int_0^1\int_0^1 \tilde p\Big[m(t,\tilde p,\tilde q)-m_0^0\Big]\,df_0^1(\tilde p,\tilde q)\\
&\quad -q^*(1-\alpha_0)[m(t,p^*,q^*)-m_0^0]^2\int_0^1\int_0^1 \tilde p\,df_0^1(\tilde p,\tilde q)\\
&= I+II+III.
\end{aligned}$$

The choice of q^* assures that

$$II\le q^*(1-\alpha_0)|m(t,p^*,q^*)-m_0^0|^2\int_0^1\int_0^1 \tilde p\,df_0^1(\tilde p,\tilde q)=-III.$$

The cancellation of these two terms gives

$$\frac{\partial}{\partial t}|m(t,.)-m_0^0|^2_{|(p^*,q^*)}\le -2\varepsilon_0\alpha_0\langle p\rangle_{|q=0}|m_0^0-m(t,p^*,q^*)|^2. \tag{54}$$

Denote $V(t)=\max_{(p,q)\in \mathrm{supp}(f_0^1)} h(t;(p,q))$ with $h(t;(p,q))=|m(t,p,q)-m_0^0|^2$, which in t is a C^1 function since m is C^1 in t. Moreover, by (53) it holds that $|\partial_t h(t;(p,q))|\le C$. Now the envelope Theorem 3.3 applies to obtain that V is absolutely continuous with derivative

$$V'(t)=\partial_t\Big(|m(t,q^*)-m_0^0|^2\Big)\qquad a.e..$$

Thus, in view of (54),

$$V'(t)\le -2\varepsilon_0\alpha_0\langle p\rangle_{|q=0}V(t)$$

and as a result

$$V(t)\le V(0)e^{-2\varepsilon_0\alpha_0\langle p\rangle_{|q=0}t},$$

which completes the proof. □

We are now in position to accomplish the proof of Theorem 3.2. The previous Step ensures that for any $t\ge 0$ and any $(p,q)\in \mathrm{supp}\ f_0^1(p,q)dpdq$,

$$W_1\Big(\delta_{m(t,p,q)},\delta_{m_0^0}\Big)=|m(t,p,q)-m_0^0|\le 2e^{-\varepsilon_0\alpha_0\langle p\rangle_{|q=0}t}.$$

According to (49) and noticing that $\langle p\rangle\ge \alpha_0\langle p\rangle_{|q=0}$, we infer that

$$W_1\Big(f^1_{t|(p,q)},\delta_{m_0^0}\Big)\le 2e^{-\varepsilon_0\alpha_0\langle p\rangle_{|q=0}t}+2e^{-\varepsilon_0\langle p\rangle t}\le 4e^{-\varepsilon_0\alpha_0\langle p\rangle_{|q=0}t}.$$

We now claim that

$$W_1\Big(f_t^1,\delta_{m_0^0}\otimes f_0^1(p,q)dpdq\Big)\le 4e^{-\varepsilon_0\alpha_0\langle p\rangle_{|q=0}t}.$$

Indeed let $\psi : K \to \mathbb{R}$ be 1-Lipschitz. Then,

$$\int_K \psi\left(df_t^1 - \delta_{m_0^0} \otimes f_0^1(p,q)dpdq\right)$$
$$= \int_0^1 \int_0^1 \Big(\int_{-1}^1 \psi(w,p,q)\,(df_{t|(p,q)} - \delta_{m_0^0}) \Big)\, df_0(p,q).$$

Since $\psi(.,p,q)$ es 1-Lipschitz, the inner integral is bounded above by $W_1(f_{t|(p,q)}, \delta_{m_0^0})$, which implies that

$$\begin{aligned}\int_K \psi\left(df_t^1 - \delta_{m_0^0} \otimes f_0^1(p,q)dpdq\right) &\leq 4e^{-\varepsilon_0\alpha_0\langle p\rangle_{|q=0}t} \int_0^1\int_0^1 df_0(p,q) \\ &= 4e^{-\varepsilon_0\alpha_0\langle p\rangle_{|q=0}t}.\end{aligned}$$

The claim follows taking supremum over all functions ψ 1-Lipschitz. The proof of the theorem is now complete. □

4. Computational experiments

We close the paper with some agent based simulations.

The numerical experiment considers a population of $N = 10000$ agents with $\alpha_0 = 60\%$ stubborn agents. We take such a high proportion to speed up the computations in view of (39) whereas it does not change the value of m_∞.

Initially,

- each non-stubborn agent has opinion chosen uniformly at random in $[0.3; 1]$, parameter q is chosen uniformly at random in $[0.2; 1]$ and we set $p = 1 - q$,
- one third of the stubborn agents has $p = 0.6$ and opinion chosen at random in $[-0.8; -0.6]$ uniformly, whereas the others have $p = 0.2$ and opinion chosen uniformly at random in $[0.4; 0.8]$.

Notice in particular that

$$\begin{aligned}\langle p\rangle_{q=0} &= \frac{1}{3}\times 0.6 + \frac{2}{3}\times 0.2 = \frac{1}{3}, \\ \langle p\rangle &= \alpha_0\langle p\rangle_{q=0} + (1-\alpha_0)\int p\,df_0^1 = 0.6 * \frac{1}{3} + 0.4 * (1-0.6) \\ &= 0.36, \\ \int pw\,df_0^0(w,p) &= \frac{1}{3}\times 0.6\times(-0.7) + \frac{2}{3}\times 0.2\times 0.6.\end{aligned}$$

It follows that

$$m_\infty = \int \frac{pw}{\langle p\rangle_{q=0}}\,df_0^0(w,p) \approx -0.18.$$

We then let the agents interact following the rules with $\gamma = 0.01$. We depict in the figures in Table 4 the density of (w,q) parameters among the non-stubborn population for different times in gray scale (the whiter is the graphic, the higher is the density). This picture clearly reveals the convergence of the density of opinion $f_{t|q}(w)dw$ among the population with q towards its mean value $m_q(t)$ (at a faster pace for higher q as predicted), and then the displacement of the curve-like density to a vertical segment located at $w \approx -0.18$. This is in complete agreement with the theoretical value m_∞ given above.

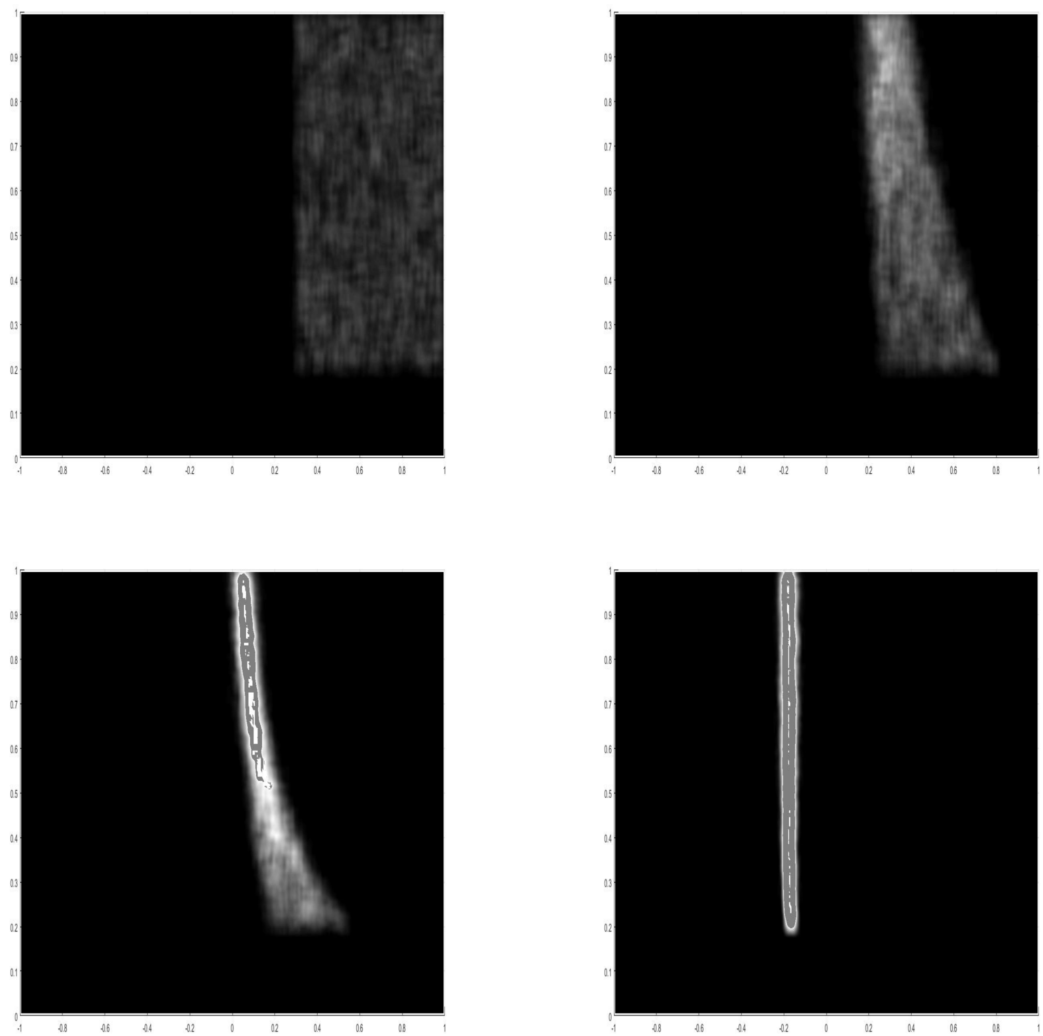

Table 1. From left to right and top to bottom, the figures represent the density of (w, q) parameters among the non-stubborn population in gray scale (the whiter is the graphic, the higher is the density) with opinion w in the horizontal axe, and q in the vertical axe after respectively 0,37,96,700 $\times$50000 interactions. The initial values of (w, p, q) are those given in the text.

Appendix A. Existence of a unique solution to the Boltzmann equation. Proof of Theorem 2.1

The existence of a unique solution to the Boltzmann equation results as an application of the classical Banach Fixed Point Theorem, as it is sketched in [15]. We provide here a detailed proof, for the reader's convenience.

Proof. Let us first introduce some notations. If $f, g \in M(K)$ are given measures, we define a finite measure $Q(f,g)$ by

$$\begin{aligned}(Q(f,g),\phi) &= \frac{1}{2}\int_{K^2\times B}(\phi(\varpi')-\phi(\varpi))\,df(\varpi)dg(\varpi_*)d\theta(\eta)\\ &\quad+\frac{1}{2}\int_{K^2\times B}(\phi(\varpi')-\phi(\varpi))\,dg(\varpi)df(\varpi_*)d\theta(\eta),\end{aligned} \tag{55}$$

for any $\phi \in C(K)$. Notice that

$$|(Q(f,g),\phi)| \leq 2\|\phi\|_\infty \|f\|\|g\|,$$

where $\|f\|$ and $\|g\|$ denote the total variation norm (1) of f and g, respectively. Consequently, the total variation norm of the measure $Q(f,g)$ verifies that

$$\|Q(f,g)\| \leq 2\|f\|\|g\|. \tag{56}$$

Observe for future use that $Q(f,f) - Q(g,g) = Q(f+g, f-g)$ which yields

$$\|Q(f,f) - Q(g,g)\| \leq 2\|f+g\|\|f-g\|. \tag{57}$$

Fix some $T > 0$ to be chosen later on. Denote by $C_T := C([0,T], P(K))$ the space of functions from $[0,T]$ with values in $P(K)$ being continuous for the total variation norm. The estimate (56) ensures that for all $f, g \in C_T$, it holds $\|Q(f_s,g_s)\| \leq C$ for any $s \in [0,T]$. Thus $\int_0^t \|Q(f_s,g_s)\|\,ds$ is finite and we can consider the integral $\int_0^t Q(f_s,g_s)\,ds$ in the Bochner sense. Since $P(K)$ is separable (because K is compact), the integration is indeed understood in the Pettis sense. In particular,

$$\left(\int_0^t Q(f_s,g_s)\,ds, \phi\right) = \int_0^t (Q(f_s,g_s),\phi)\,ds, \qquad \text{for any } \phi \in C(K). \tag{58}$$

Under these notations, (7) can be rewritten as

$$\int_K \phi\,df_t = \int_K \phi\,df_0 + \int_0^t (Q(f_s,f_s),\phi)\,ds,$$

i.e.

$$f_t = f_0 + \int_0^t Q(f_s,f_s)\,ds =: J(f)(t). \tag{59}$$

Our purpose is to find a fixed point of J in the closed subspace X_T of C_T defined as

$$X_T := \{f \in C_T : f(0) = f_0 \text{ and } \max_{0\leq t\leq T}\|f_t\| \leq 2\|f_0\|\},$$

where T is sufficiently small. We endow X_T with the sup-norm given by $\|f\|_{X_T} = \max_{0\le t\le T}\|f_t\|$. For any $f \in X_T$, a direct application of the Dominated Convergence Theorem ensures that $J(f) \in C_T$. Moreover, in view of (56), we get

$$\begin{aligned}\|J(f)(t)\| &\le \|f_0\| + \int_0^t \|Q(f_s,f_s)\|\,ds \le \|f_0\| + 2T\max_{0\le t\le T}\|f_s\|^2 \\ &\le \|f_0\| + 8T\|f_0\|^2.\end{aligned}$$

Taking $T \le 1/(8\|f_0\|)$ guarantees that $J(f) \in X_T$. Next we prove that J is in fact a strict contraction. Recall that by (57) we know

$$\begin{aligned}\|J(f)(t) - J(g)(t)\| &\le \int_0^t \|Q(f_s,f_s) - Q(g_s,g_s)\|\,ds \\ &\le 2\int_0^t \|f_s + g_s\|\|f_s - g_s\|\,ds \\ &\le 8T\|f_0\|\|f-g\|.\end{aligned}$$

The choice e.g. $T = 1/(16\|f_0\|)$ provides that $\|J(f) - J(g)\| \le \frac{1}{2}\|f-g\|$. The existence of a unique fixed point of J in X_T consequently follows.

Taking $\phi = 1$ in (7) and recalling that $f_0 \in P(K)$ shows that $\int_K f_t(\varpi) = 1$. It just remains to see that $f_t \ge 0$ to infer that $f_t \in P(K)$ with $\|f_t\| = \|f_0\| = 1$. At this point, we could then repeat the previous argument to extend f_t to $[T, 2T]$, $[2T, 3T]$, and so on, and conclude the existence proof. Proposition A.1 below is devoted to prove the non-negativity of f, which completes this proof. □

Remark A.1. *Bearing in mind that f is continuous in time, it is no difficult to see that $\int_K \phi(w)df_t(w)$ is a C^1 function with respect to t, whose derivative is specified by (6).*

Indeed, with the notations introduced in the previous proof, it holds that

$$\begin{aligned}\left\|\frac{f_{t+h}-f_t}{h} - Q(f_t,f_t)\right\| &= \left\|\frac{1}{h}\int_t^{t+h} Q(f_s,f_s)\,ds - Q(f_t,f_t)\right\| \\ &\le \frac{1}{h}\int_t^{t+h}\left\|Q(f_s,f_s) - Q(f_t,f_t)\right\|ds.\end{aligned}$$

Thanks to (57) we infer that

$$\left\|\frac{f_{t+h}-f_t}{h} - Q(f_t,f_t)\right\| \le 8\|f_0\|\frac{1}{h}\int_t^{t+h}\|f_s - f_t\|\,ds,$$

which goes to 0 as $h \to 0$, since f is continuous. Therefore, (7) can be rewritten as

$$\partial_t f = Q(f,f).$$

We complete the proof of Theorem 2.1 showing the uniqueness and non-negativity of f_t.

Proposition A.1. *Let $g_0 \in P(K)$ and $\lambda \geq 1$. There exists a unique $g \in C^1([0,+\infty), M_+(K))$ such that $g_{|t=0} = g_0$ and for $t > 0$ solving*

$$\partial_t g_t + \lambda g_t = Q(g_t, g_t) + \lambda g_t \int_K dg_t. \tag{60}$$

Remark A.2. *Notice that $\|f_t\| = 1$ guarantees that f_t is a solution to (60). By uniqueness f_t must belong to $M_+(K)$, hence is nonnegative.*

Proof. We begin introducing some definitions. Let $\Gamma : M(K) \times M(K) \to M(K)$ be a measure determined by $\Gamma(f,g) = Q(f,g) + \frac{\lambda}{2}(g \int f + f \int g)$. Denote $\Gamma(f) := \Gamma(f,f)$ and $Q(f) = Q(f,f)$. In view of (57), $\Gamma(f)$ is continuous in f with respect to the total variation norm.

Moreover, we claim that $\Gamma(f,g) \geq 0$ if f and g are non-negative. To see this, note that the measure Q can be represented by $Q(f,g) = Q_+(f,g) - Q_-(f,g)$ with

$$(Q_+(f,g), \phi) = \frac{1}{2} \int_{K^2 \times B} \phi(\varpi') \Big(df(\varpi) dg(\varpi_*) + dg(\varpi) df(\varpi_*) \Big) d\theta(\eta) \tag{61}$$

and

$$Q_-(f,g) = \frac{1}{2} \left(f \int_{K \times B} dg(\varpi) d\theta(\eta) + g \int_{K \times B} df(\varpi) d\theta(\eta) \right), \tag{62}$$

for any $\phi \in C(K)$.

Then, Γ can be expressed as

$$\Gamma(f,g) = Q_+(f,g) + \frac{1}{2}(\lambda - 1) \left(f \int_{K \times B} dg(\varpi) d\theta(\eta) + g \int_{K \times B} df(\varpi) d\theta(\eta) \right) \geq 0,$$

because $\lambda \geq 1$ and the claim follows.

Furthermore, whenever $g \geq f \geq 0$,

$$\Gamma(g) \geq \Gamma(f) \geq 0. \tag{63}$$

Indeed, since $g + f$ and $g - f$ are non-negative measures,

$$\Gamma(g) - \Gamma(f) = \Gamma(g + f, g - f) \geq 0.$$

We need to find $g \in C([0,+\infty, M_+(K))$ such that $\int g \leq 1$ and

$$g_t = e^{-\mu t} g_0 + \int_0^t e^{-\mu(t-s)} \Gamma(g_s)\, ds. \tag{64}$$

It will be obtained as the limit of the following sequence $g^n : [0,+\infty) \to M(K)$, $n \geq 0$, defined iteratively by $g^0 = 0$ and

$$g_t^n = e^{-\lambda t} g_0 + \int_0^t e^{-\lambda(t-s)} \Gamma(g_s^{n-1})\, ds. \tag{65}$$

Since $g_0 \geq 0$ and the measure Γ is continuous, non-negative and non-decreasing, it is easy to see that $g_t^n \geq g_t^{n-1} \geq 0$ for any n and $t > 0$. Clearly, $g^n \in C([0,+\infty), M(K))$.

Even more, by integrating equation (64) in K, taking into account that $\int Q(f) = 0$ for any $f \in M(K)$, we deduce that the total mass $g_t^n(K) = \int_K dg_t^n$ satisfies

$$g_t^n(K) = e^{-\lambda t} + \lambda \int_0^t e^{-\lambda(t-s)} \Big(g_s^{n-1}(K)\Big)^2 \, ds.$$

By induction, $g_t^n(K) \leq 1$ for any t. Therefore, for any non-negative $\phi \in C(K)$, the sequence $(\int_K \phi \, dg_t^n)_n$ is non-decreasing and bounded. It ensures the existence of a limit $(g_t, \phi) := \lim_{n\to\infty} \int \phi \, dg_t^n$.

The estimate (56) yields that $\|\Gamma(g_t^n)\| \leq 2 + \lambda$ uniformly in n, t. Then for any $T > 0$, it follows then that

$$\|g_t^n - g_s^n\| \leq C(T)|s - t|,$$

for any $s, t \in [0, T]$ and any n. Applying Arzela-Ascoli theorem, we have, up to a subsequence, that $g^n \to g$ in $C_{loc}([0, +\infty), M^+(K))$, which implies that $g_t(K) \leq 1$ and $g \in C_{loc}([0, +\infty), M_+(K))$. Passing to the limit in (65), we get that g satisfies (64).

Observe that the continuity of Γ guarantees that g belongs in fact to C^1.

Eventually, if $\tilde{g}$ is another solution of (64), then, by (57),

$$\|g_t - \tilde{g}_t\| \leq \int_0^t e^{-\mu(t-s)} \|\Gamma(g_s) - \Gamma(\tilde{g}_s)\| \, ds \leq C \int_0^t e^{-\mu(t-s)} \|g_s - \tilde{g}_s\| \, ds,$$

so that $\|g_t - \tilde{g}_t\| = 0$ by Gronwall's Lemma. As a result $g = \tilde{g}$ and the proof is finished. $\square$

Appendix B. Grazing Limit

We perform exhaustively the passage to the grazing limit, considering all of the possible balances between the transport and the diffusion terms. Our proof is based on the arguments given in [47], adapted to our specific model.

Thm B.1. *In the interaction rule* (4) *admit that* $\sigma^2 = \gamma\lambda$ *for some* $\lambda > 0$. *Given an initial condition* $f_0 \in P(K)$, *consider the solution,* f, *of the Boltzmann-like equation* (6) *given by Theorem 2.1. If* $f_\gamma(\tau) := f(t)$, *stands for the time rescaled probability density according to* $\tau = \gamma t$, *it holds, up to subsequences, that* $f_\gamma \to g$ *as* $\gamma \to 0$ *in* $C([0, T], P(K))$ *for any* $T > 0$. *Furthermore, the limit* $g \in C([0, +\infty), P(K))$ *satisfies for any* $\tau \geq 0$ *and any* $\phi \in C^\infty(K)$,

$$\begin{aligned}\int_K \phi \, dg_\tau = \int_K \phi \, df_0 &+ \int_0^\tau \int_K (m(\tau) - w)\langle p\rangle q \phi_w(\varpi) \, dg_s(\varpi) ds \\ &+ \frac{\lambda}{2} \int_0^\tau \int_K q^2 D^2(|w|) \phi_{ww}(\varpi) \, dg_s(\varpi) ds.\end{aligned} \tag{66}$$

Moreover, if $\frac{\sigma^2}{\gamma} = \lambda \to 0$, *then* $g \in C([0, +\infty), P(K))$ *verifies the transport equation*

$$\int_K \phi \, dg_\tau = \int_K \phi \, df_0 + \int_0^\tau \int_K (m(\tau) - w)\langle p\rangle q \phi_w(\varpi) \, dg_s(\varpi) ds. \tag{67}$$

If conversely, $\frac{\sigma^2}{\gamma} \to +\infty$*, rescaling time as* $\tau := \gamma^\alpha t$*, for some* $\alpha \in (0,1)$*, it holds that* $f_\gamma \to g$ *as* $\gamma \to 0$*, where* g *is determined by*

$$\int_K \phi \, dg_\tau = \int_K \phi \, df_0 + \frac{\lambda}{2} \int_0^\tau \int_{K^2} \phi_{ww}(\varpi) q^2 D^2(|w|) \, dg_\tau(\varpi), \tag{68}$$

being $\lambda > 0$ *now such that* $\sigma^2 = \lambda \gamma^\alpha$*.*

Proof. First of all consider the case $\sigma^2 = \gamma\lambda$ for some $\lambda > 0$. Let $\phi \in C^3(K)$. The rescaled measure $f_\gamma(\tau)$ solves

$$\begin{aligned}
&\frac{d}{d\tau} \int_K \phi(\varpi) f_\gamma(\tau, \varpi) d\varpi \\
&= \frac{1}{\gamma} \int_B \int_{K^2} (\phi(\varpi') - \phi(\varpi)) f_\gamma(\tau, \varpi) f_\gamma(\tau, \varpi_*) d\varpi d\varpi_* d\theta(\eta).
\end{aligned}$$

Recall that $\varpi = (w, p, q)$ and $\varpi' = (w', p, q)$. We perform a Taylor expansion of ϕ (with respect to the w variable) up to second order:

$$\phi(\varpi') - \phi(\varpi) = \phi_w(\varpi)(w' - w) + \frac{1}{2} \phi_{ww}(\tilde{\varpi})(w' - w)^2,$$

with $\tilde{\varpi} = (\tilde{w}, p, q)$ being $\tilde{w} = \theta w + (1-\theta) w'$ for some $\theta \in (0,1)$. Note that $\int_B \eta \Theta(\eta) d\eta = 0$ and $\int_B \eta^2 \Theta(\eta) d\eta = \sigma^2$. Then substituting this expansion into the previous equation and using the updating rules (4), it yields

$$\begin{aligned}
&\frac{d}{d\tau} \int_K \phi(\varpi) f_\gamma(\tau, \varpi) d\varpi \\
&= \int_{K^2} \phi_w(\varpi) p_* q (w_* - w) f_\gamma(\tau, \varpi) f_\gamma(\tau, \varpi_*) d\varpi d\varpi_* \\
&\quad + \frac{\sigma^2}{2\gamma} \int_K \phi_{ww}(\varpi) q^2 D^2(|w|) f_\gamma(\tau, \varpi) d\varpi \\
&\quad + \frac{\gamma}{2} \int_{K^2} \phi_{ww}(\varpi) p_*^2 q^2 (w_* - w)^2 f_\gamma(\tau, \varpi) f_\gamma(\tau, \varpi_*) d\varpi d\varpi_* + R(\tau, \gamma, \sigma),
\end{aligned} \tag{69}$$

where

$$\begin{aligned}
R(\tau, \gamma, \sigma) = \frac{1}{2\gamma} \int_B \int_{K^2} &\Theta(\eta) \Big(\gamma p_* q (w_* - w) + \eta q D(|w|) \Big)^2 \\
&(\phi_{ww}(\tilde{\varpi}) - \phi_{ww}(\varpi)) f_\gamma(\tau, \varpi) f_\gamma(\tau, \varpi_*) d\varpi d\varpi_* d\eta.
\end{aligned}$$

Observe that the first integral in (69) can be written as follows

$$\begin{aligned}
I &= \int_K q \phi_w(\varpi) f_\gamma(\tau, \varpi) d\varpi \int_K p_* w_* f_\gamma(\tau, \varpi_*) d\varpi_* \\
&\quad - \int_K q w \phi_w(\varpi) f_\gamma(\tau, \varpi) d\varpi \int_K p_* f_\gamma(\tau, \varpi_*) d\varpi_* \\
&= \int_K (\langle wp \rangle - \langle p \rangle w) q \phi_w(\varpi) f_\gamma(\tau, \varpi) d\varpi \\
&= \int_K (m_\gamma(\tau) - w) \langle p \rangle q \phi_w(\varpi) f_\gamma(\tau, \varpi) d\varpi,
\end{aligned}$$

where $m_\gamma(\tau) := \frac{1}{\langle p\rangle}\int_K wpf_\gamma(\tau,\varpi)d\varpi$. The previous analysis implies that

$$\begin{aligned}
&\frac{d}{d\tau}\int_K \phi(\varpi)f_\gamma(\tau,\varpi)d\varpi \\
&= \int_K (m_\gamma(\tau)-w)\langle p\rangle q\phi_w(\varpi)f_\gamma(\tau,\varpi)d\varpi \\
&\quad +\frac{\sigma^2}{2\gamma}\int_K q^2D^2(|w|)\phi_{ww}(\varpi)f_\gamma(\tau,\varpi)d\varpi \\
&\quad +\frac{\gamma}{2}\int_{K^2}\phi_{ww}(\varpi)p_*^2q^2(w_*-w)^2f_\gamma(\tau,\varpi)f_\gamma(\tau,\varpi_*)d\varpi d\varpi_* + R(\tau,\gamma,\sigma),
\end{aligned}$$

which integrated in time gives,

$$\begin{aligned}
&\int_K \phi(\varpi)(f_\gamma(\tau',\varpi)-f_\gamma(\tau,\varpi))d\varpi \\
&= \int_\tau^{\tau'}\int_K (m_\gamma(\tau)-w)\langle p\rangle q\phi_w(\varpi)f_\gamma(s,\varpi)d\varpi ds \\
&\quad +\frac{\sigma^2}{2\gamma}\int_\tau^{\tau'}\int_K q^2D^2(|w|)\phi_{ww}(\varpi)f_\gamma(s,\varpi)d\varpi ds \\
&\quad +\frac{\gamma}{2}\int_\tau^{\tau'}\int_{K^2}\phi_{ww}(\varpi)p_*^2q^2(w_*-w)^2f_\gamma(s,\varpi)f_\gamma(s,\varpi_*)d\varpi d\varpi_* ds \\
&\quad +\int_\tau^{\tau'} R(s,\gamma,\sigma)ds.
\end{aligned} \tag{70}$$

We now show that, whenever σ^2/γ remains bounded as $\gamma,\sigma\to 0$, then

$$\lim_{\gamma,\sigma\to 0} R(\tau,\gamma,\sigma)\to 0 \qquad \text{uniformly in } \tau\in\mathbb{R}. \tag{71}$$

Using that $\varpi=(w,p,q)$, $\varpi'=(w',p,q)$ and $|\tilde{w}-w|=(1-\theta)|w'-w|\le|w'-w|$, we easily see that

$$|\phi_{ww}(\tilde{\varpi})-\phi_{ww}(\varpi)|\le\|\phi_{www}\|_\infty|\tilde{w}-w|\le\|\phi_{www}\|_\infty|w'-w|.$$

As a result,

$$\begin{aligned}
|R(\tau,\gamma,\sigma)| \le \frac{\|\phi_{www}\|_\infty}{2\gamma}\int_B\int_{K^2}\Theta(\eta)\Big|\gamma p_*q(w_*-w) \\
+\eta qD(|w|)\Big|^3 f_\gamma(\tau,\varpi)f_\gamma(\tau,\varpi_*)d\varpi d\varpi_* d\eta.
\end{aligned}$$

Applying the inequality $(a+b)^3\le 8(a/2+b/2)^3\le 4(a^3+b^3)$ and taking into account that $p_*,q,\gamma,D(|w|)\in[0,1]$ and $w,w_*\in[-1,1]$, we deduce that

$$|\gamma p_*q(w_*-w)+\eta qD(|w|)|^3\le 4(|\gamma p_*q(w_*-w)|^3+|\eta qD(|w|)|^3)\le 32\gamma^3+4\eta^3.$$

Consequently,

$$|R(\tau,\gamma,\sigma)| = \|\phi_{www}\|_\infty\left(16\gamma^2+\frac{2\sigma^2}{\gamma}\sigma E[|Y|^3]\right). \tag{72}$$

and the limit (71) follows since we assumed $E[|Y|^3]<\infty$.

We denote $X = C^3(K)$ with the usual norm $\|\phi\|_X = \sum_{|\alpha|\le 3} \|\partial^\alpha \phi\|_\infty$. Recall that $p, q, p^*, q^*, D(|w|) \in [0,1]$, $f_\gamma(\tau,\cdot) \in P([-1,1])$ for all τ and $m_\gamma(t), w, w^* \in [-1,1]$. Invoking (70) and (72) it can be inferred that

$$\begin{aligned}
&\left|\int_K \phi(\varpi)(f_\gamma(\tau,\varpi) - f_\gamma(\tau',\varpi))d\varpi\right| \\
&\quad \le \left[2\|\phi_w\|_\infty + \|\phi_{ww}\|_\infty \left(2\gamma + \frac{\sigma^2}{\gamma}\right)\right. \\
&\qquad \left. + \|\phi_{www}\|_\infty \left(16\gamma^2 + \frac{2\sigma^2}{\gamma}\sigma E[|Y|^3]\right)\right](\tau' - \tau) \\
&\quad \le \|\phi\|_X \left(2 + 2\gamma + 16\gamma^2 + \frac{2\sigma^2}{\gamma}\sigma E[|Y|^3] + \frac{\sigma^2}{\gamma}\right)(\tau' - \tau). \\
&\quad = C\|\phi\|_X(\tau' - \tau).
\end{aligned}$$

Taking supremum gives

$$\sup_{\phi\in X, \|\phi\|_X \le 1} \left|\int_K \phi(\varpi)(f_\gamma(\tau,\varpi) - f_\gamma(\tau',\varpi))d\varpi\right| \le C(\tau' - \tau). \tag{73}$$

Define

$$\|\mu\| := \sup_{\phi\in X, \|\phi\|_X \le 1} \int_K \phi\, d\mu. \tag{74}$$

Then, (73) can be read as

$$\|f_\gamma(\tau) - f_\gamma(\tau')\| \le C|\tau' - \tau|,$$

for any $\gamma \in [0,1]$ and any $\tau, \tau' \in [0,+\infty)$, where the constant C is independent of γ, τ, τ'. It can be shown that the norm in (74) induces the weak topology on $P(K)$ (see Ref.[17], Lemma 5.3 and, Corollary 5.5). We have thus shown that the sequence of continuous probability measure valued functions $f_\gamma : [0,+\infty) \to P(K)$ are uniformly equicontinuous. In addition, $\|f_\gamma(\tau)\| \le 1$ for any τ and γ, hence Arzela-Ascoli theorem, together with a diagonal argument, ensure the existence of $g \in C([0,\infty); P(K))$ and a subsequence $(\gamma_n)_n$ converging to 0 such that $f_{\gamma_n} \to g$ in $C([0,T]; P(K))$ for any $T > 0$.

It remains to pass to the limit in (70). Since the norm in (74) metrizes the weak convergence, it is well known (see for example [50]) that

$$\max_{\tau\in[0,T]} \|f_{\gamma_n}(\tau) - g(\tau)\| \to 0 \quad \text{as } n \to \infty,$$

can be expressed in terms of the Wasserstein distance as

$$\lim_{n\to\infty} \max_{\tau\in[0,T]} W_1(f_{\gamma_n}(\tau), g(\tau)) = 0. \tag{75}$$

We can rewrite (75) as

$$\int_K \varphi(\varpi) f_{\gamma_n}(\tau,\varpi)d\varpi \to \int_K \varphi(\varpi) g(\tau,\varpi)d\varpi, \tag{76}$$

uniformly on compacts $0 \le \tau \le T$ for any $T > 0$ and for any Lipschitz function φ. As a result we have

$$m_\gamma(\tau) = \frac{1}{\langle p\rangle}\int_K wpf_\gamma(\tau,\varpi)d\varpi \to \frac{1}{\langle p\rangle}\int_K wpg(\tau,\varpi)d\varpi =: m(\tau),$$

uniformly for $\tau \in [0,T]$, $T > 0$. Passing to the limit in (70) this shows that for any $\phi \in C^3(K)$ and any $\tau' \ge \tau \ge 0$,

$$\begin{aligned}\int_K \phi\, dg_{\tau'} = \int_K \phi\, dg_\tau &+ \int_\tau^{\tau'}\int_K (m(\tau) - w)\langle p\rangle q\phi_w(\varpi)\, dg_s(\varpi)ds\\ &+ \frac{\lambda}{2}\int_\tau^{\tau'}\int_K q^2D^2(|w|)\phi_{ww}(\varpi)\, dg_s(\varpi)ds,\end{aligned}$$

which proves (66) as desired.

Admit now that $\frac{\sigma^2}{\gamma} \to 0$. Taking limit as $\lambda \to 0$ in (70) shows (67).

Finally, suppose that $\sigma^2 = \lambda\gamma^\alpha$ for some $\lambda > 0$ and $\alpha \in (0,1)$. In particular $\frac{\sigma^2}{\gamma} \to +\infty$ as $\gamma \to 0$, hence the diffusion dominates the transport. Rescaling time as $\tau := \gamma^\alpha t$, (69) now reads as

$$\begin{aligned}&\frac{d}{d\tau}\int_K \phi(\varpi)f_\gamma(\tau,\varpi)d\varpi\\ &= \gamma^{1-\alpha}\int_{K^2}\phi_w(\varpi)p_*q(w_* - w)f_\gamma(\tau,\varpi)f_\gamma(\tau,\varpi_*)d\varpi d\varpi_*\\ &\quad + \frac{\lambda}{2}\int_K \phi_{ww}(\varpi)q^2D^2(|w|)f_\gamma(\tau,\varpi)d\varpi\\ &\quad + \frac{\gamma^{2-\alpha}}{2}\int_{K^2}\phi_{ww}(\varpi)p_*^2q^2(w_* - w)^2f_\gamma(\tau,\varpi)f_\gamma(\tau,\varpi_*)d\varpi d\varpi_* + \tilde{R}(\tau,\gamma,\sigma),\end{aligned}$$

where $\tilde{R}(\tau,\gamma,\sigma) = \gamma^{3-\alpha}R(\tau,\gamma,\sigma)$. Using (72), we have

$$|\tilde{R}(\tau,\gamma,\sigma)| \le \|\phi\|_X(16\gamma^{5-\alpha} + 2\lambda\gamma^2\sigma E|Y|^3) = o(1)\|\phi\|_X.$$

Arguing as before it can be shown that the limit g satisfies (68), and the proof is complete.

□

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