

COUNTABLE CONTRACTION MAPPINGS IN METRIC SPACES: INVARIANT SETS AND MEASURES

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ABSTRACT. We consider a complete metric space (X, d) and a countable number of contraction mappings on X , $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$. We show the existence of a *smallest* invariant set (with respect to inclusion) for \mathcal{F} . If the maps F_i are of the form $F_i(x) = r_i x + b_i$ on $X = \mathbb{R}^d$, we can prove a converse of the classic result on contraction mappings. Precisely, we can show that for that case, there exists a *unique* bounded invariant set if and only if $r = \sup_i r_i$ is strictly smaller than 1.

Further, if $\rho = \{\rho_k\}_{k \in \mathbb{N}}$ is a probability sequence, we show that if there exists an invariant measure for the system (\mathcal{F}, ρ) , then its support must be precisely this smallest invariant set. If in addition there exists any *bounded* invariant set, this invariant measure is unique - even though there may be more than one invariant set.

1. INTRODUCTION

A map F from a metric space (X, d) into itself is a **contraction**, if there exists a constant c , $0 < c < 1$, such that $d(F(x), F(y)) \leq cd(x, y)$, for all $x, y \in X$. We denote by $\text{Lip}(F)$ the smallest of all such constants and we call it *Lipschitz constant* or *contraction factor* of F .

In [6], Hutchinson introduced the notion of *invariant set* and *invariant measure* for a finite set of contraction mappings from a complete metric space (X, d) into itself. In particular in that paper he proved the now *classical* results:

Theorem 1.1. [6] *Let $\mathcal{F} = \{F_1, \dots, F_N\}$ be a finite family of contraction mappings in the complete metric space (X, d) . Let $\mathcal{C}(X)$ be the set of non-empty closed and bounded subsets of X , and let $\rho_1, \dots, \rho_N \in (0, 1)$ and $\sum_{i=1}^N \rho_i = 1$. Then we have:*

- *There exists a unique set $K \in \mathcal{C}(X)$ that is invariant with respect to \mathcal{F} , i.e.*

$$K = \overline{\bigcup_{i=1}^N F_i(K)}.$$

K is in fact compact and is the closure of the set of fixed points of all finite compositions of elements of \mathcal{F} .

- *There exists a unique Borel regular (outer) measure μ with bounded support, and total mass 1, that is invariant with respect to (\mathcal{F}, ρ) , i.e.*

$$\mu = \sum_{i=1}^N \rho_i F_{i\#} \mu,$$

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where $F_{i\sharp}\mu$ is the measure defined by $F_{i\sharp}\mu(E) = \mu(F_i^{-1}(E))$ for each $E \subset X$.

- The support of μ , $\text{supp } \mu$ is precisely the invariant set K .

Similar results can be found in [2]. For general references see [3], [4] and [8].

If instead of a finite number of contraction mappings one considers a countable collection, the notions of invariant set and invariant measure can be extended in a natural way:

Definition 1.2. Let $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$ be a countable family of contraction mappings in the complete metric space (X, d) . We say that a non-empty set $E \subset X$ is an **invariant set** for \mathcal{F} if

$$E = \overline{\bigcup_{i=1}^{\infty} F_i(E)}.$$

If $\rho = (\rho_1, \rho_2, \dots)$ is a probability sequence, i.e. $\rho_i \in (0, 1)$ and $\sum_{i=1}^{\infty} \rho_i = 1$, we say that an outer measure μ is an **invariant measure** for (\mathcal{F}, ρ) if

$$\mu = \sum_{i=1}^{\infty} \rho_i F_{i\sharp}\mu,$$

where $F_{i\sharp}\mu$ is (as before) the measure defined by $F_{i\sharp}\mu(E) = \mu(F_i^{-1}(E))$ for each $E \subset X$.

Finite families of contraction mappings automatically satisfy two conditions which allow to ensure the existence and uniqueness of a bounded invariant set: on one hand the boundedness of the set of fixed points, and on the other hand the fact that the maximum of Lipschitz constants is *strictly* less than 1. In general, if one has a countable system these conditions are not automatically satisfied.

Definition 1.3. A set $\mathcal{F} = \{F_i\}_{i \in I}$, for $I \subset \mathbb{N}$ either finite or infinite, in a complete metric space (X, d) , where F_i are contraction mappings, is called Iterated Function System (IFS). We denote by r the supremum of the contraction factors, and by D the set of fixed points, i.e.

$$r := \sup_{i \in I} \{r_i : r_i \text{ contraction factor of } F_i\} \quad \text{and}$$

$$D := \{x_i, i \in I : F_i(x_i) = x_i\}.$$

In the case that D is bounded and $r < 1$, Bandt [1] showed the existence and uniqueness of a **bounded invariant set** for \mathcal{F} , where $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$ is a countable family of contraction mappings. Countable iterated function systems were first introduced by Mauldin and Williams ([11], see also [9] and [10]).

In the present article we show that for **any** countable family of contraction mappings \mathcal{F} there exists an invariant set (see also [13]). In fact, there exists a smallest invariant set, with respect to inclusion, for \mathcal{F} . We show that this set is the closure of the set of fixed points of finite compositions of members of \mathcal{F} . For this result we do not need to assume that D is bounded neither that $r < 1$.

It follows that the bounded invariant set obtained by Bandt is - as in the finite case - the closure of the set of fixed points of finite compositions of members of \mathcal{F} .

We further show that the boundedness of D is necessary for the existence of a **bounded** invariant set. In fact, since any invariant set contains D , if there exists a bounded invariant set then D is bounded. However we show that the condition $r < 1$ is not necessary for the existence of a bounded invariant set.

Further, the condition that D be bounded is not sufficient: we exhibit an example in which the set of fixed points is bounded, but there does not exist a bounded

invariant set. However, if $X = \mathbb{R}$ and the system only contains non-decreasing functions, the boundedness of D does suffice (cf. Theorem 2.7).

In addition, we prove a kind of converse to the Theorem by Bandt in [1]: under certain restrictions, if there exists a unique bounded invariant set, then necessarily $r < 1$ (cf. Theorem 2.10).

Finally, we prove that the support of **any** invariant measure for the countable IFS (\mathcal{F}, ρ) , where ρ is a probability sequence, must coincide precisely with the smallest invariant set that we showed to exist. We further show, that if there exists a bounded invariant set, then the invariant measure exists and is **unique**, even though the invariant set might not be unique.

2. INVARIANT SETS

Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contraction mappings in the complete metric space (X, d) . First, we will prove that there exists a smallest invariant set for \mathcal{F} , with respect to inclusion: the closure of the set of fixed points of finite compositions of members of \mathcal{F} .

As before, if $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ is a countable family of contraction mappings in the complete metric space (X, d) , x_i will denote the fixed point of F_i and D will be the set of fixed points of members of \mathcal{F} . For all $i \in \mathbb{N}$, r_i will denote the Lipschitz constant of F_i , i.e., $r_i := \text{Lip}(F_i)$, and $r := \sup_i r_i$. Further, $F_{i_1 \dots i_k}$ will denote the composition $F_{i_1} \circ \dots \circ F_{i_k}$, $x_{i_1 \dots i_k}$ will be the fixed point of $F_{i_1 \dots i_k}$ and P will be the set of fixed points of finite compositions of members of \mathcal{F} .

We first need the following result, which is analogous to the finite IFS case:

Lemma 2.1. *Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contraction mappings in the complete metric space (X, d) and let P be the set of fixed points of finite compositions of members of \mathcal{F} . If $A \subset X$ is a non-empty closed set such that $F_i(A) \subset A$ for all $i \in \mathbb{N}$, then $P \subset A$.*

Proof.

Let $x_{i_1 \dots i_p}$ be the fixed point of $F_{i_1 \dots i_p}$, and let $a \in A$. Then

$$\lim_{k \rightarrow \infty} F_{i_1 \dots i_p}^k(a) = x_{i_1 \dots i_p}.$$

Since $F_i(A) \subset A$ para todo $i \in \mathbb{N}$, $F_{i_1 \dots i_p}(a) \in A$. Then we have $F_{i_1 \dots i_p}^k(a) \in A$ for all k .

Since A is closed, $x_{i_1 \dots i_p} \in A$. Hence, $P \subset A$. \square

Theorem 2.2. *Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contraction mappings in the complete metric space (X, d) . If P is the set of fixed points of finite compositions of members of \mathcal{F} , then \overline{P} is the smallest invariant set for \mathcal{F} , with respect to inclusion.*

Proof. First, we will prove that \overline{P} is an invariant set for \mathcal{F} .

Note that for a fixed $N \in \mathbb{N}$, if we consider the finite sub-family $\{F_1, \dots, F_N\}$, and let P_N be the set of fixed points of finite compositions of F_i with $1 \leq i \leq N$, from Theorem 1.1 it follows that $\overline{P_N}$ is the unique compact invariant set for the IFS $\{F_1, \dots, F_N\}$.

In order to prove the inclusion $\bigcup F_i(\overline{P}) \subset \overline{P}$ it is enough to show that $F_i(P) \subset \overline{P}$ for all i . For this, let $i \in \mathbb{N}$ be fixed, and let $x_{\alpha_1 \dots \alpha_n} \in P$ be the fixed point of $F_{\alpha_1 \dots \alpha_n}$. We need to show that $F_i(x_{\alpha_1 \dots \alpha_n}) \in \overline{P}$.

We define the following sequence $\{y_k\}_{k \in \mathbb{N}}$ in P : for each k , let y_k be the fixed point of $F_i \circ F_{\alpha_1 \dots \alpha_n}^k$. So $y_1 = x_{i\alpha_1 \dots \alpha_n}$, $y_2 = x_{i\alpha_1 \dots \alpha_n \alpha_1 \dots \alpha_n}$; etc.

If $N := \max\{\alpha_1, \dots, \alpha_n, i\}$, then $x_{\alpha_1 \dots \alpha_n} \in P_N$ and $y_k \in P_N$ for all k . Therefore,

$$d(y_k, F_i(x_{\alpha_1 \dots \alpha_n})) \leq r_i d(F_{\alpha_1 \dots \alpha_n}^k(y_k), x_{\alpha_1 \dots \alpha_n}) < (r_{\alpha_1} \dots r_{\alpha_n})^k \text{diam } P_N.$$

Hence, since $r_{\alpha_1} \dots r_{\alpha_n} < 1$ and $\text{diam } P_N < \infty$, we have

$$\lim_{k \rightarrow \infty} y_k = F_i(x_{\alpha_1 \dots \alpha_n}),$$

which implies that $F_i(x_{\alpha_1 \dots \alpha_n}) \in \bar{P}$, as we wanted to show.

For the other inclusion we will show that $P \subset \bigcup F_i(\bar{P})$. Let $x_{\alpha_1 \dots \alpha_n} \in P$ and consider the sequence $\{z_k\}_{k \in \mathbb{N}}$, where z_k is the fixed point of the composition $F_{\alpha_2 \dots \alpha_n} \circ F_{\alpha_1 \dots \alpha_n}^k$. As before, if $N := \max\{\alpha_1, \dots, \alpha_n\}$ then $x_{\alpha_1 \dots \alpha_n} \in P_N$ and $z_k \in P_N$ for all k . Consequently,

$$\begin{aligned} d(z_k, F_{\alpha_2 \dots \alpha_n}(x_{\alpha_1 \dots \alpha_n})) &\leq r_{\alpha_2} \dots r_{\alpha_n} (r_{\alpha_1} \dots r_{\alpha_n})^k d(z_k, x_{\alpha_1 \dots \alpha_n}) \\ &< (r_{\alpha_1} \dots r_{\alpha_n})^k \text{diam } P_N, \end{aligned}$$

which implies $\lim_{k \rightarrow \infty} z_k = F_{\alpha_2 \dots \alpha_n}(x_{\alpha_1 \dots \alpha_n}) \in \bar{P}$ and therefore

$$x_{\alpha_1 \dots \alpha_n} = F_{\alpha_1}(F_{\alpha_2 \dots \alpha_n}(x_{\alpha_1 \dots \alpha_n})) \in F_{\alpha_1}(\bar{P}).$$

Thus, the closure of P is an invariant set for \mathcal{F} .

In order to show that \bar{P} is the smallest invariant set, let A be an invariant set for \mathcal{F} . By Definition 1.2 A is non-empty and closed and satisfies $F_i(A) \subset A$ for all $i \in \mathbb{N}$. By Lemma 2.1 we obtain that $P \subset A$, therefore $\bar{P} \subset A$. \square

Remark 2.3. Notice that in the previous theorem we do not assume that D is bounded neither that $r < 1$. Therefore, for all countable family of contraction mappings there exists an invariant set.

Remark 2.4. The assertion that \bar{P} is an invariant set can also be obtained using Hutchinson's Theorem 1.1 and a result in [13].

Recalling the results of Bandt [1], which show that if $r < 1$ and D is bounded, there exists a unique closed and **bounded invariant set** with respect to \mathcal{F} ; using Theorem 2.2, this unique set must necessarily coincide with \bar{P} (see also [12] and [5]). Note that Bandt also shows that this set is not necessarily compact. This extends completely the result of Hutchinson to the countable IFS case:

Corollary 2.5. Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contraction mappings in the complete metric space (X, d) . If $r = \sup_{i \in \mathbb{N}} \text{Lip}(F_i) < 1$ and D , the set of fixed points of elements of \mathcal{F} , is bounded, then \bar{P} is the unique bounded invariant set for \mathcal{F} , where P is the set of fixed points of finite compositions of members of \mathcal{F} .

From Theorem 2.2 it follows that the boundedness of D is a necessary condition for the existence of a bounded invariant set. Indeed, since every invariant set contains D , if there exists a bounded invariant set, then D must be bounded.

On the other hand, the next example shows that the condition $r < 1$ is not necessary.

Example 2.6. For every $i \in \mathbb{N}$ we define $F_i : \mathbb{R} \rightarrow \mathbb{R}$,

$$F_i(x) = \left(\frac{i}{i+1} \right) x + \frac{1}{(i+1)^2}.$$

Then, the set of fixed points $D = \{1/(i+1)\}_{i \in \mathbb{N}}$ is bounded, but $\sup_i r_i = 1$. However, there exists a bounded invariant set, for example, $[0, 1/2]$. Indeed, for every $a \leq 0$ and $b \geq 1/2$, the closed interval $[a, b]$ is an invariant set.

The previous example can be extended to every countable family of non-decreasing contraction mappings in \mathbb{R} :

Theorem 2.7. *Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contraction mappings in \mathbb{R} , such that every F_i is non-decreasing. If the set of fixed points of members of \mathcal{F} is bounded, then there exists a bounded invariant set for \mathcal{F} .*

Proof. Let D be the set of fixed points of members of \mathcal{F} , and let us consider $\alpha = \inf D$ and $\beta = \sup D$. We will show that the interval $I = [\alpha, \beta]$ satisfies $F_i(I) \subset I$ for all $i \in \mathbb{N}$.

Since every F_i is continuous and non-decreasing, we have that $F_i(I) = [F_i(\alpha), F_i(\beta)]$. Further, since F_i is contractive, $F_i(\beta) - F_i(x_i) < \beta - x_i$ and $F_i(x_i) - F_i(\alpha) < x_i - \alpha$. Therefore $\alpha < F_i(\alpha) \leq F_i(\beta) < \beta$. Thus, $F_i(I) \subset I$.

The conclusion now follows from Lemma 2.1. \square

However, in the general case, the boundedness of D is not a sufficient condition for the existence of a bounded invariant set, since it can be the case (as we show below) that D is bounded, but P is not.

Example 2.8. We consider the contraction mappings in \mathbb{R} defined by:

$$F_i(x) = -\frac{i}{i+1}x + \frac{2i+1}{i}; \quad \tilde{F}_i(x) = -\frac{i}{i+1}x + \frac{1}{i+1}, \quad i \in \mathbb{N}.$$

Let \mathcal{F} be the countable family $\mathcal{F} := \{F_i : i \in \mathbb{N}\} \cup \{\tilde{F}_i : i \in \mathbb{N}\}$. The set of fixed points is contained in $[0, 2]$. However, there does not exist any bounded invariant set for \mathcal{F} .

To see this, we consider the compositions $\tilde{F}_i \circ F_i$ and $F_i \circ \tilde{F}_i$ ($i \in \mathbb{N}$) and look at the set $\{y_i : \tilde{F}_i \circ F_i(y_i) = y_i\} \cup \{z_i : F_i \circ \tilde{F}_i(z_i) = z_i\}$. A simple computation shows that the fixed point of $\tilde{F}_i \circ F_i$ is $y_i = -\frac{2i(i+1)}{2i+1}$ and the fixed point of $F_i \circ \tilde{F}_i$ is $z_i = \frac{-i^2 + (i+1)^2(2i+1)}{i(2i+1)}$. Thus, the set P can not be bounded. \square

We will conclude this section proving a kind of converse to the Theorem by Bandt. In some cases we will be able to prove that, if there exists a unique bounded invariant set, then necessarily $r < 1$. We begin by proving a general lemma about enlargements of an invariant set A . Recall that the ε -enlargement of a set A in a metric space (X, d) is defined by:

$$A_\varepsilon := \{y \in X : d(y, A) < \varepsilon\}.$$

Lemma 2.9. *Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contraction mappings in a complete metric space (X, d) , and let A an invariant set for \mathcal{F} . If $\alpha > 0$, then $F_i(A_\alpha) \subset A_\alpha$ for all i .*

Proof. Let $x \in A_\alpha$. Then, by definition of A_α , there is $y \in A$ such that $d(x, y) < \alpha$. Since A is invariant, $F_i(y) \in A$. Moreover, $d(F_i(x), F_i(y)) \leq r_i \cdot d(x, y) < \alpha$ and therefore $F_i(x) \in A_\alpha$. \square

For $X = \mathbb{R}$, and F_i similarities, we can sharpen the previous result:

Theorem 2.10. *Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contractive similarities in \mathbb{R} ($|F_i(x) - F_i(y)| = r_i|x - y|$ for all $x, y \in \mathbb{R}$, $r_i < 1$), such that $r = \sup_i r_i = 1$. If there exists a bounded set A that is invariant for \mathcal{F} , then there exists $\alpha > 0$ such that $\overline{A_\alpha}$ is invariant. (Hence if a bounded invariant set exists, it is not unique, in contrast to the case $r < 1$!)*

Proof. Let $\alpha > 0$ be such that $A_\alpha = (a, b) = I$ for some interval I . By the previous Lemma, we have that $F_i(I) \subset I$ for each i . Then $\overline{\bigcup_{i=1}^{\infty} F_i(I)} \subset \overline{I}$.

For the other inclusion, let $x \in I$ and let $\delta = \max\{b - x, x - a\}$. Now choose $r_i > \delta/(b - a)$, which is possible, since $r = 1$. Since each F_i is a similarity, $F_i(I)$ is an interval either $(F_i(a), F_i(b))$ or $(F_i(b), F_i(a))$ depending on the monotonicity of F_i .

If F_i is increasing, $F_i(b) - F_i(a) = r_i(b - a) > \delta$. This implies that

$$F_i(b) - F_i(a) > b - x \quad \text{and} \quad F_i(b) - F_i(a) > x - a,$$

and consequently

$$x - F_i(a) > b - F_i(b) > 0 \quad \text{and} \quad F_i(b) - x > F_i(a) - a > 0.$$

Hence $F_i(a) < x < F_i(b)$ and so $x \in F_i(I)$.

If in turn F_i is decreasing, an analogous reasoning allows us to conclude that, $x \in F_i(I)$.

$$\text{Hence } \bar{I} \subset \overline{\bigcup_{i=1}^{\infty} F_i(\bar{I})}.$$

□

As a corollary we have a converse to the Theorem of Bandt:

Corollary 2.11. *Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contractive similarities in \mathbb{R} . If there exists a **unique** bounded invariant set for \mathcal{F} , then $r = \sup_i \text{Lip}(F_i) < 1$.*

The previous theorem can be extended to contractive similarities in \mathbb{R}^n that are multiples of the identity, i.e. $F_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F_i(\mathbf{x}) = r_i \mathbf{x} + \mathbf{b}_i$, $|r_i| < 1$.

Theorem 2.12. *Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$, with $F_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F_i(\mathbf{x}) = r_i \mathbf{x} + \mathbf{b}_i$, $|r_i| < 1$. Let $r = \sup |r_i|$, and P be the subset of \mathbb{R}^n of fixed points of finite compositions of $\{F_i\}_{i \in \mathbb{N}}$. If P is bounded and $r = 1$, then there exists a rectangle $R = I_1 \times \cdots \times I_n$ such that $P \subsetneq R$ that is invariant for \mathcal{F} .*

Proof. We write $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{b}_i = (b_{i1}, \dots, b_{in})$, thus

$$F_i(\mathbf{x}) = r_i \mathbf{x} + \mathbf{b}_i = (r_i x_1 + b_{i1}, \dots, r_i x_n + b_{in}),$$

and we call f_{ij} the maps from \mathbb{R} to \mathbb{R} defined by the j -th coordinate ($f_{ij}(x_j) = r_i x_j + b_{ij}$), so $F_i(\mathbf{x}) = (f_{i1}(x_1), \dots, f_{in}(x_n))$ and f_j is a contractive similarity in \mathbb{R} , of contraction factor r_i , for $j = 1, \dots, n$.

We have that \mathbf{x} is a fixed point for F_i if and only if x_j is a fixed point for f_{ij} for $j = 1, \dots, n$.

Now, for every $j = 1, \dots, n$, let $\mathcal{F}_j := \{f_{ij}\}_{i \in \mathbb{N}}$ be the countable IFS on the line, defined by the "coordinate" maps of \mathcal{F} . If P_j is the set of fixed points of finite compositions of the maps from \mathcal{F}_j , by our assumption we have that \bar{P}_j is bounded and invariant for \mathcal{F}_j for $j = 1, \dots, n$. By the proof of Theorem 2.10, there exist intervals I_j such that

- $\bar{P}_j \subsetneq I_j$, and therefore $P \subsetneq R := I_1 \times \cdots \times I_n$.
- $f_{ij}(I_j) \subset I_j$ for all i .

Hence, since $F_i = r_i Id + \mathbf{b}_i$ we have that $F_i(R) = f_{i1}(I_1) \times \cdots \times f_{in}(I_n) \subset I_1 \times \cdots \times I_n = R$ for all i .

Let now $\mathbf{x} \in R$ (i.e. $x_j \in I_j$, $1 \leq j \leq n$). If $I_j = [\alpha_j, \beta_j]$ we take $\delta_j := \max\{\beta_j - x_j; x_j - \alpha_j\}$. Since $r = 1$, we choose $i \in \mathbb{N}$ such that $r_i > \max\{\frac{\delta_j}{\beta_j - \alpha_j} : 1 \leq j \leq n\}$.

From the proof of Theorem 2.10 it follows that for each j , $1 \leq j \leq n$, $x_j \in f_{ij}(I_j)$. Hence $\mathbf{x} \in f_{i1}(I_1) \times \cdots \times f_{in}(I_n) = F_i(R)$.

So R is invariant for \mathcal{F} .

□

We again have the same corollary:

Corollary 2.13. *Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contractive similarities in \mathbb{R}^n , such that $F_i(\mathbf{x}) = r_i \mathbf{x} + \mathbf{b}_i$, $|r_i| < 1$. If there exists a **unique** bounded invariant set for \mathcal{F} , then $r = \sup |r_i| < 1$.*

3. INVARIANT MEASURES

In addition to providing a complete proof of Hutchinson's theorem for the most general case, Kravchenko in [7] generalizes Hutchinson's theorem 1.1 to the case of a countable set of maps and he gives a sufficient condition for the existence and uniqueness of an invariant measure. We first need to recall the following definition.

Definition 3.1. *A measure ν is separable if there exists a separable Borel set $A \subset X$ such that $\nu(X \setminus A) = 0$.*

Note that if ν is a finite measure, ν is separable if and only if $\nu(X \setminus \text{supp } \nu) = 0$.

Theorem 3.2. [7] *Let (X, d) be a complete metric space. Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contraction mappings with fixed points x_i . Let $\rho = \{\rho_i\}_{i \in \mathbb{N}}$, be a probability sequence, i.e. $0 < \rho_i < 1$ and $\sum_{i=1}^{\infty} \rho_i = 1$. If $\sum_{i=1}^{\infty} \rho_i d(x_1, x_i) < \infty$, then there exists a unique measure $\mu \in M_s(X)$ that is invariant with respect to (\mathcal{F}, ρ) , i.e.*

$$\mu = \sum_{i \in \mathbb{N}} \rho_i F_{i\#} \mu.$$

Here $M_s(X)$ is the space of all separable probability measures that satisfy $\int f d\mu < \infty$ for all $f : X \rightarrow \mathbb{R}$ with finite Lipschitz constant.

Note that the hypothesis of this theorem are slightly weaker than the ones of Bandt, since r is not required to be strictly smaller than 1. Moreover, if D is bounded and $\sum_{i=1}^{\infty} \rho_i = 1$ we have

$$\sum_i \rho_i d(x_1, x_i) \leq \sum_i \rho_i \text{diam}(D) = \text{diam}(D) < \infty.$$

Hence, in the case of bounded D , for each (countable) probability sequence we have a unique measure in $M_s(X)$ that is invariant with respect to (\mathcal{F}, ρ) , (independently of the value of r).

Remark 3.3. *In [12] Mihail and Miculescu worked with Infinite Iterated Function System with the same hypothesis as Bandt in [1] from a different viewpoint. They showed that for these IIFS, the unique invariant set for \mathcal{F} (which is bounded because of the hypothesis on the IIFS) coincides with the closure of the canonical projection of the shift space.*

In this case, analogously to the finite case, one has immediately that the unique invariant measure is $\pi_{\#} \tau$, and that the support of this measure is the unique invariant set for \mathcal{F} (here τ is the product measure on $\mathbb{N}^{\mathbb{N}}$ induced by $\rho(i) = \rho_i$ on each factor).

Their results rely strongly on the fact that the set of fixed points is bounded and the supremum of the Lipschitz constants of the system is strictly smaller than 1.

We will prove next that the support of any invariant measure for (\mathcal{F}, ρ) must coincide with the smallest invariant set for \mathcal{F} , where ρ is a probability sequence, even for the case $r = 1$.

We start proving that if μ is an invariant measure for (\mathcal{F}, ρ) then its support is an invariant set for \mathcal{F} . We first need the following result:

Lemma 3.4. *Let (X, d) be a metric space. If $F : X \rightarrow X$ is a Lipschitz map and μ is a measure in X , then $F(\text{supp } \mu) \subset \text{supp } F_{\#} \mu$.*

Proof.

Let us consider $y = F(x)$ for some $x \in \text{supp } \mu$. In order to prove that $y \in \text{supp } F_{\#} \mu$, we need to show that any ball centered at y has positive $F_{\#} \mu$ -measure.

We take $\varepsilon > 0$ and consider the ball $B(x, \delta) := \{z \in X : d(x, z) < \delta\}$, with $\delta = r^{-1}\varepsilon$ where $r = \text{Lip}(F)$. Since F is Lipschitz,

$$F(B(x, \delta)) \subset B(F(x), r\delta) = B(y, \varepsilon),$$

and then,

$$B(x, \delta) \subset F^{-1}(B(y, \varepsilon)).$$

Hence,

$$F_{\#}\mu(B(y, \varepsilon)) = \mu(F^{-1}(B(y, \varepsilon))) \geq \mu(B(x, \delta)) > 0,$$

because $x \in \text{supp } \mu$. Thus $y \in \text{supp } F_{\#}\mu$. \square

We are now ready to prove the announced theorem.

Theorem 3.5. *Let (X, d) be complete metric space. Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contraction mappings on X and let $\rho = \{\rho_i : i \in \mathbb{N}\}$ be a probability sequence, i.e., $0 < \rho_i < 1$ and $\sum_{i=1}^{\infty} \rho_i = 1$. If μ is an invariant measure for (\mathcal{F}, ρ) , then the support of μ is an invariant set for \mathcal{F} .*

Proof.

Assume that μ is an invariant measure for (\mathcal{F}, ρ) and let $A := \text{supp } \mu$. Then, by Definition 1.2,

$$\mu = \sum_{i=1}^{\infty} \rho_i F_{i\#}\mu,$$

where $F_{i\#}\mu$ is the measure defined by $F_{i\#}\mu(E) = \mu(F_i^{-1}(E))$ for each $E \subset X$. By Lemma 3.4 we have that $F_i(A) \subset \text{supp } F_{i\#}\mu$ for all i . Further, it is clear that

$$\text{supp } F_{i\#}\mu \subset \text{supp } \left(\sum_{i=1}^{\infty} \rho_i F_{i\#}\mu \right) = \text{supp } \mu = A.$$

Hence, $\bigcup_{i=1}^{\infty} F_i(A) \subset A$. Since A is closed, we obtain $\overline{\bigcup_{i=1}^{\infty} F_i(A)} \subset A$.

On the other hand, let $a \in A$ and $\varepsilon > 0$. Since $\mu(B(a, \varepsilon)) > 0$ and $\mu = \sum_{i=1}^{\infty} \rho_i F_{i\#}\mu$, there must exist i such that $\mu(F_i^{-1}(B(a, \varepsilon))) > 0$. Consequently

$$F_i^{-1}(B(a, \varepsilon)) \cap A \neq \emptyset.$$

That is, there exists $x \in A$ such that $d(F_i(x), a) < \varepsilon$. Then $a \in \overline{\bigcup_{i=1}^{\infty} F_i(A)}$. Thus, $A \subset \overline{\bigcup_{i=1}^{\infty} F_i(A)}$.

The proof is complete. \square

From this result and Theorem 2.2 one deduces that the support of any invariant measure for (\mathcal{F}, ρ) contains the set \overline{P} , the closure of the set of fixed points of finite compositions of members of \mathcal{F} . In the following theorem we prove that indeed the support of μ is equal to \overline{P} .

Theorem 3.6. *Let (X, d) be a complete metric space. Let $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ be a countable family of contraction mappings on X and let $\rho = \{\rho_i : i \in \mathbb{N}\}$ be a probability sequence, i.e., $0 < \rho_i < 1$ and $\sum_{i=1}^{\infty} \rho_i = 1$. If μ is an invariant measure for (\mathcal{F}, ρ) , then $\text{supp } \mu = \overline{P}$, where P the set of fixed points of finite compositions of members of \mathcal{F} .*

Proof.

From Theorems 3.5 and 2.2 we have that $\overline{P} \subset \text{supp } \mu$.

In order to prove the other inclusion, let us consider $x \notin \overline{P}$. We will prove that $x \notin \text{supp } \mu$, by showing that there exists a neighbourhood of x of zero μ -measure.

Since $x \notin \overline{P}$, let $\varepsilon = d(x, \overline{P})/2 > 0$. Let

$$G = B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$

We will prove that $\mu(G) = 0$.

Define, as before, the set \overline{P}_ε as $\overline{P}_\varepsilon = \{y \in X : d(y, \overline{P}) < \varepsilon\}$. Notice that $G \cap \overline{P}_\varepsilon = \emptyset$.

Now, let $i \in \mathbb{N}$ be arbitrary but fixed throughout the proof. Since F_i is a contraction map whose fixed point is x_i and whose contraction factor is r_i , $\{F_i^k(x)\}_{k \in \mathbb{N}}$ converges to x_i and $\{r_i^k\}_{k \in \mathbb{N}}$ converges to 0.

Then there exists $k = k(i)$ such that $r_i^k < 1/2$ and $d(F_i^k(x), x_i) < \varepsilon/2$.

Further, again by the contractivity of F_i , we have that for any $j \in \mathbb{N}$, $F_i^j(G) \subset B(F_i^j(x), r_i^j \varepsilon)$, in particular $F_i^k(G) \subset B(F_i^k(x), r_i^k \varepsilon)$.

Moreover, if $z \in B(F_i^k(x), r_i^k \varepsilon)$ then

$$d(z, \overline{P}) \leq d(z, x_i) \leq d(z, F_i^k(x)) + d(F_i^k(x), x_i) < r_i^k \varepsilon + \varepsilon/2 < \varepsilon.$$

Therefore $F_i^k(G) \subset \overline{P}_\varepsilon$. Hence, $G \subset [(F_i^k)^{-1}(\overline{P}_\varepsilon) \setminus \overline{P}_\varepsilon]$.

To finish our claim, it will be enough to prove that $\mu((F_i^k)^{-1}(\overline{P}_\varepsilon)) = \mu(\overline{P}_\varepsilon)$.

Since \overline{P} is an invariant set, we have $F_n(\overline{P}_\varepsilon) \subset \overline{P}_\varepsilon$ for all $n \in \mathbb{N}$. Consequently, for every $(i_1 \dots i_k) \in \mathbb{N}^k$, $F_{i_1 \dots i_k}(\overline{P}_\varepsilon) \subset \overline{P}_\varepsilon$ and therefore

$$\mu(\overline{P}_\varepsilon) \leq \mu(F_{i_1 \dots i_k}^{-1}(\overline{P}_\varepsilon)) \quad \text{for all } (i_1 \dots i_k) \in \mathbb{N}^k. \quad (1)$$

Note that if for some $(i_1 \dots i_k) \in \mathbb{N}^k$ we had a strict inequality in the last equation, by the invariance of μ we would have that

$$\mu(\overline{P}_\varepsilon) = \sum_{i_1 \dots i_k} \rho_{i_1} \dots \rho_{i_k} \mu(F_{i_1 \dots i_k}^{-1}(\overline{P}_\varepsilon)) \geq \mu(\overline{P}_\varepsilon).$$

Therefore from equation (1) we must have $\mu(\overline{P}_\varepsilon) = \mu(F_{i_1 \dots i_k}^{-1}(\overline{P}_\varepsilon))$ for all choices $(i_1 \dots i_k) \in \mathbb{N}^k$.

In particular, by taking $i_1 = \dots = i_k = i$, we obtain $\mu(\overline{P}_\varepsilon) = \mu((F_i^k)^{-1}(\overline{P}_\varepsilon))$, and the proof is complete. \square

As noted before, the existence of the invariant measure depends only on the relatively weak condition $\sum_i \rho_i d(x_1, x_i) < \infty$, independently of the value of r . (For example if the set of fixed points of F_i is bounded, the condition is already satisfied and guaranties the existence *and* uniqueness of an invariant measure).

Our result shows, that in contrast to the case of *invariant sets* which may not be unique, if an invariant measure having bounded support exists, it is unique. Indeed, if μ is an invariant measure whose support is bounded, from our result it follows that the set of fixed points of members of \mathcal{F} is bounded, which implies that $\sum_i \rho_i d(x_1, x_i) < \infty$ and, consequently we obtain the uniqueness of the invariant measure.

REFERENCES

- [1] Bandt C., Self-similar sets. I. Topological Markov chains and mixed self-similar sets, Math. Nachr., 1989, **142**, 107–123.
- [2] Barnsley M. F., Demko S., Iterated function systems and the global construction of fractals, Proc. Roy. Soc. London Ser. A, 1985, **399**(1817), 243–275.
- [3] Falconer K. J., The Geometry of Fractal Sets, Cambridge Tracts in Mathematics, 85, Cambridge Univ. Press, Cambridge, 1986.
- [4] Falconer K. J., Fractal Geometry. Mathematical Foundations and Applications, John Wiley and Sons, Chichester, 1990.
- [5] Hille M. R., Remarks on limit sets of infinite iterated function systems, Monatsh. Math., 2012, **168**(2), 215–237.
- [6] Hutchinson J. E., Fractals and self-similarity, Indiana Univ. Math. J., 1981, **30**(5), 713–747.
- [7] Kravchenko A. S., Completeness of the space of separable measures in the Kantovorich-Rubinshtein metric, Siberian Math. J., 2006, **47**(1), 68–76.
- [8] Mattila P., Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability, Cambridge Stud. Adv. Math., 44, Cambridge Univ. Press, Cambridge, 1995.

- [9] Mauldin R. D., Infinite iterated function systems: theory and applications, *Progr. Probab.*, 1995, **37**, 91–110.
- [10] Mauldin R. D., Urbański M., Dimensions and measures in infinite iterated function systems, *Proc. London Math. Soc.*(3), 1996, **73**(1), 105–154
- [11] Mauldin R. D., Williams S. C., Random recursive constructions: asymptotic geometric and topological properties, *Trans. Amer. Math. Soc.*, 1986, **295**(1), 325–346.
- [12] Mihail A. and Miculescu R., The shift space for an infinite iterated function system, *Math. Rep. (Bucur.)*, 2009, **11(61)**(1), 21–32.
- [13] Secelean N. A., The existence of the attractor of countable iterated function systems, *Mediterr. J. Math.*, 2012, **9**(1), 61-79.

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