EXTENDED BEST POLYNOMIAL APPROXIMATION OPERATOR IN ORLICZ SPACES

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Abstract. In this paper we consider the best polynomial approximation operator, defined in an Orlicz space $L^\Phi(B)$, and its extension to $L^\varphi(B)$, where φ is the derivative function of Φ. A characterization of these operators and several properties are obtained.

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1. Introduction

In this paper we set $\Im$ for the class of all continuous and nondecreasing functions φ defined for all real number $t \geq 0$, with $\varphi(0^+) = 0$, $\varphi(t) \to \infty$ as $t \to \infty$ and $\varphi(t) > 0$ for $x > 0$. We also assume a Δ_2 condition for the functions φ, which means that there exists a constant $\Lambda = \Lambda_\varphi > 0$ such that $\varphi(2a) \leq \Lambda\varphi(a)$ for all $a \geq 0$.

Now given $\varphi \in \Im$ we consider $\Phi(x) = \int_0^x \varphi(t)\,dt$. Observe that $\Phi : [0, \infty) \to [0, \infty)$ is a convex function such that $\Phi(a) = 0$ iff $a = 0$. For such a function Φ we have $\frac{\Phi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\Phi(x)}{x} \to \infty$ as $x \to \infty$, and according to [4], a function with this property is called an N function. Observe that the function φ satisfies a Δ_2 condition if and only if the function Φ satisfies a Δ_2 condition.

If $\varphi \in \Im$ then it satisfies a Δ_2 condition. Thus the next inequality holds

$$\frac{1}{2}(\varphi(a) + \varphi(b)) \leq \varphi(a + b) \leq \Lambda_\varphi(\varphi(a) + \varphi(b)) \tag{1.1}$$

for every $a, b \geq 0$.

Also note that the Δ_2 condition on Φ implies

$$\frac{x}{2\Lambda_\varphi}\varphi(x) \leq \Phi(x) \leq x\varphi(x), \tag{1.2}$$

for every $x \geq 0$.

Let B be a bounded measurable set in $\mathbb{R}^n$. If $\varphi \in \Im$, we denote by $L^\varphi(B)$ the class of all Lebesgue measurable functions f defined on $\mathbb{R}^n$ such that $\int_B \varphi(t|f|)\,dx < \infty$ for some $t > 0$ and where dx denotes

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the Lebesgue measure on $\mathbb{R}^n$. Note that as $\varphi \in \Im$ and it satisfies a Δ_2 condition then $L^\varphi(B)$ is the space of all measurable functions f defined on $\mathbb{R}^n$ such that $\int_B \varphi(|f|)\,dx < \infty$. For the convex function Φ, $L^\Phi(B)$ is the classical Orlicz space very well studied in [4] and [5].

Let Π^m be the space of algebraic polynomials, defined on $\mathbb{R}^n$, of degree at most m. Then a polynomial $P \in \Pi^m$ is called a best approximation of $f \in L^\Phi(B)$ if and only if

$$\int_B \Phi(|f-P|)\,dx = \inf_{Q\in\Pi^m} \int_B \Phi(|f-Q|)\,dx. \tag{1.3}$$

Definition 1. *For $f \in L^\Phi(B)$ we set $\mu_\Phi(f)$ for the set of all polynomials P that satisfy (1.3).*

In the sequel we also refer to $\mu_\Phi(f)$ as the multivalued operator defined for functions in $L^\Phi(B)$ and images on Π^m.

In this paper we study the nature of this best polynomial approximation for functions in $L^\Phi(B)$ and we extend, in a continuous way, the definition of best polynomial approximation for functions belonging to $L^\varphi(B)$ where $\varphi = \Phi'$. These results extend those obtained in [1] for the L^p case. In Section 2 we define the best polynomial approximation operator for each $f \in L^\Phi(B)$ and we characterize this best approximation in a similar way as it has done in [3] for functions of $L^\Phi(B)$ in the case that the approximation class is a lattice instead of the space of polynomials. We also get a strong type inequality for $f \in L^\Phi(B)$ which generalizes Theorem 2.1 in [2] where the extended best polynomial approximation operator is considered for functions in $L^p(B)$. In Section 3 we use this inequality to extend the best polynomial approximation from $L^\Phi(B)$ to $L^\varphi(B)$, where $\varphi = \Phi'$. This is done in an easier way than the one developed in [1], where the existence of the extension is proved without using the inequality in Theorem 2.4. At the end of this section, we prove the uniqueness and a continuity property for the extended best polynomial approximation of $f \in L^\varphi(B)$ for a strictly increasing functions $\varphi \in \Im$.

2. Existence and uniqueness of the best polynomial approximation operator in $L^\Phi(B)$

For $P \in \Pi^m$ we set $\|P\|_\infty = \max_{x\in B} |P(x)|$ and $\|P\|_1 = \int_B |P|\,dx$. We begin with the existence of the best polynomial approximation operator of functions in $L^\Phi(B)$. We start with the next lemma.

Lemma 2.1. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let P_n be a sequence in Π^m, such that there exists a constant C that satisfies $\int_B \Phi(|P_n|)\,dx \le C$. Then, the sequence P_n is uniformly bounded.*

Proof. From Jensen's inequality we have

$$|B|\Phi\Big(\frac{1}{|B|}\int_B |P_n|\,dx\Big) \le \int_B \Phi(|P_n|)\,dx \le C. \tag{2.1}$$

Then, since $\|P\|_1$ is equivalent to $\|P\|_\infty$, for $P \in \Pi^m$ and using the Δ_2 condition on Φ, we obtain

$$\Phi(\|P_n\|_\infty) \leq M,$$

for some constant M. Then, as $\Phi(x)$ goes to ∞ when x goes to ∞ the lemma follows. □

The next two theorems follow standard techniques. However, for the sake of completeness, detailed proofs of them are included.

Theorem 2.2. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let $f \in L^\Phi(B)$. Then, there exists $P \in \Pi^m$ such that*

$$\int_B \Phi(|f-P|)\,dx = \inf_{Q\in\Pi^m} \int_B \Phi(|f-Q|)\,dx.$$

Proof. Let $I = \inf_{Q\in\Pi^m} \int_B \Phi(|f-Q|)\,dx$, then there exists a sequence $\{P_n\}_{n\in\mathbb{N}} \subset \Pi^m$ such that

$$\int_B \Phi(|f-P_n|)\,dx \to I \text{ as } n \to \infty. \tag{2.2}$$

Due to the monotonicity and convexity of Φ on $[0,\infty)$, we get

$$\Phi\left(\frac{|P_n|}{2}\right) \leq \Phi\left(\frac{1}{2}|P_n - f| + \frac{|f|}{2}\right) \leq \frac{1}{2}\Phi(|P_n - f|) + \frac{1}{2}\Phi(|f|).$$

Thus

$$\int_B \Phi\left(\frac{|P_n|}{2}\right) dx \leq \frac{1}{2}\int_B \Phi(|P_n - f|)\,dx + \frac{1}{2}\int_B \Phi(|f|)\,dx,$$

and then

$$2\int_B \Phi\left(\frac{|P_n|}{2}\right) dx \leq \int_B \Phi(|f|)\,dx + I + 1. \tag{2.3}$$

Now, Lema 2.1 implies $\|P_n\|_\infty \leq K$. Hence, there exists a subsequence $\{P_{n_k}\} \subseteq \{P_n\}_{\{n\in\mathbb{N}\}}$ such that $\{P_{n_k}\}$ converges uniformly on Π^m.

Let $P = \lim_{n_k\to\infty} P_{n_k}$. Since Φ satisfies the Δ_2 condition we have

$$\Phi(|f - P_{n_k}|) \leq \Lambda_\Phi(\Phi(|f|) + \Phi(|P_{n_k}|)) \leq \Lambda_\Phi(\Phi(|f|) + \Phi(K)).$$

Then, by Lebesgue Dominated Convergence Theorem, we have $I = \int_B \Phi(|f-P|)\,dx$. □

The next theorem gives a characterization of the best polynomial approximation of functions in $L^\Phi(B)$.

Theorem 2.3. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let $f \in L^\Phi(B)$. Then $P \in \Pi^m$ is in $\mu_\Phi(f)$ if and only if*

$$\int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx = 0, \tag{2.4}$$

for every $Q \in \Pi^m$.

Proof. For P in $\mu_\Phi(f)$ and $Q \in \Pi^m$ we set

$$F_Q(\varepsilon) = \int_B \Phi(|f - P + \varepsilon Q|)\, dx.$$

Next we prove that F_Q is a convex function defined on $[0, \infty)$. For $a, b \geq 0$ such that $a + b = 1$, we have

$$F_Q(a\epsilon_1 + b\epsilon_2) = \int_B \Phi(|(a+b)(f-P) + (a\epsilon_1 + b\epsilon_2)Q|)\, dx \leq$$
$$\int_B \Phi(a|(f-P) + \epsilon_1 Q| + b|(f-P) + \epsilon_2 Q|)\, dx \leq$$
$$\int_B a\Phi(|(f-P)| + \epsilon_1 Q|)\, dx + \int_B b\Phi(|(f-P)| + \epsilon_2 Q|)\, dx =$$
$$aF_Q(\epsilon_1) + bF_Q(\epsilon_2),$$

for every $\epsilon_1,\ \epsilon_2 \geq 0$. Then

$$F_Q(0) = \min_{[0,\infty)} F_Q(\epsilon), \tag{2.5}$$

and this identity holds if and only if $0 \leq F_Q'(0^+)$.

Now, using the Mean Value Theorem we have

$$\frac{|\Phi(|f - P + \epsilon Q|) - \Phi(|f - P|)|}{\epsilon |Q|} \leq |Q|(\varphi(|f-P|) + \varphi(|Q|)),$$

for $0 \leq \epsilon \leq 1$.

Then, since $|Q|(\varphi(|f-P|) + \varphi(|Q|))$ is an integrable function, we are allowed to differentiate inside the integral in the formula of $F_Q(\epsilon)$ and therefore

$$0 \leq F_Q'(0^+) = \int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\, dx, \tag{2.6}$$

for any $Q \in \Pi^m$.

Now for any polynomial $Q \in \Pi^m$ we take the polynomial $-Q$ in (2.6) and this completes the proof. □

The following result, similar to Theorem 2.1 in [2], provides us an inequality that we will need below.

Theorem 2.4. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\, dt$ and let $f \in L^\varphi(B)$. Suppose the polynomial $P \in \Pi^m$ satisfies*

$$\int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\, dx = 0, \tag{2.7}$$

for every $Q \in \Pi^m$. Then

$$\int_B \varphi(|P|)|Q|\, dx \leq 5\Lambda_\varphi \int_B \varphi(|f|)|Q|\, dx, \tag{2.8}$$

for every $Q \in \Pi^m$ satisfying $\mathrm{sgn}(Q(t)P(t)) = (-1)^\eta$ *at any $t \in B$ such that $Q(t)P(t) \neq 0$ and where $\eta = 0$ or $\eta = 1$.*

Proof. Suppose first let $Q \in \Pi^m$ such that $Q(t)P(t) > 0$.

Let $N = \{t \in B : f(t) > P(t)\}$ and $L = \{t \in B : f(t) \leq P(t)\}$.
Then

$$0 = \int_{N \cup L} \varphi(|f - P|) \operatorname{sgn}(f - P) Q \, dx =$$

$$\int_N \varphi(|f - P|) \operatorname{sgn}(f - P) Q \, dx + \int_L \varphi(|f - P|) \operatorname{sgn}(f - P) Q \, dx$$

Thus

$$\int_N \varphi(|f - P|) \, Q \, dx = \int_L \varphi(|f - P|) \, Q \, dx. \tag{2.9}$$

Let $H(t) = \varphi(|P(t) - f(t)|)Q(t)$ and consider the sets
$U_1 = N \cap \{t \in B : P(t) \geq 0\}$, $U_2 = N \cap \{t \in B : P(t) < 0\}$,
$U_3 = L \cap \{t \in B : P(t) \geq 0\}$, $U_4 = L \cap \{t \in B : P(t) < 0\}$.
Then, by (2.9), we get

$$\int_{U_1 \cup U2} H \, dx = \int_{U_3 \cup U4} H \, dx$$

and therefore

$$\int_{U_1} H \, dx - \int_{U_4} H \, dx = \int_{U_3} H \, dx - \int_{U_2} H \, dx. \tag{2.10}$$

Due to the monotonicity of φ, we have

$$\int_B \varphi(|P|)|Q| \, dx \leq \int_B \varphi(|P - f| + |f|)|Q| \, dx$$

and using (1.1) we get

$$\int_B \varphi(|P - f| + |f|)|Q| \, dx \leq \Lambda_\varphi \int_B \varphi(|P - f|)|Q| \, dx + \Lambda_\varphi \int_B \varphi(|f|)|Q| \, dx =$$

$$\Lambda_\varphi \int_{\bigcup_{i=1}^{4} U_i} |H| \, dx + \Lambda_\varphi \int_B \varphi(|f|)|Q| \, dx =$$

$$\Lambda_\varphi \sum_{i=1}^{4} \int_{U_i} |H| \, dx + \Lambda_\varphi \int_B \varphi(|f|)|Q| \, dx = \Lambda_\varphi (I_1 + I_2)$$

Now, we will find an upper bound of $I_1 = \sum_{i=1}^{4} \int_{U_i} |H| \, dx$.

Note that we have $|P - f| \leq |f|$ on U_1 and U_4. Next, since the monotonicity of φ, we obtain

$$\begin{aligned} \int_{U_1 \cup U_4} |H| \, dx &= \int_{U_1} |H| \, dx + \int_{U_4} |H| \, dx \leq \\ \int_{U_1} \varphi(|f|)|Q| \, dx + \int_{U_4} \varphi(|f|)|Q| \, dx &\leq 2 \int_B \varphi(|f|)|Q| \, dx. \end{aligned} \tag{2.11}$$

Since $\text{sgn}Q = \text{sgn}P$, from (2.10) and (2.11), we get

$$\begin{aligned}\int_{U_2} |H|\,dx + \int_{U_3} |H|\,dx = \int_{U_2} (-H)\,dx + \int_{U_3} H\,dx = \\ \int_{U_1} H\,dx - \int_{U_4} H\,dx = \int_{U_1} |H|\,dx + \int_{U_4} |H|\,dx = \\ \int_{U_1\cup U_4} |H|\,dx \leq 2\int_B \varphi(|f|)|Q|\,dx\end{aligned} \tag{2.12}$$

Therefore $I_1 \leq 4\int_B \varphi(|f|)|Q|\,dx$ and

$$\int_B \varphi(|P|)|Q|\,dx \leq 5\Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx. \tag{2.13}$$

Now if $Q \in \Pi^m$ satisfies $Q(t)P(t) < 0$ we proceed in an analogous way to obtain (2.12), then

$$\begin{aligned}&\int_{U_2} |H|\,dx + \int_{U_3} |H|\,dx = \int_{U_2} H\,dx - \int_{U_3} H\,dx = \\ &-\int_{U_1} H\,dx + \int_{U_4} H\,dx = \int_{U_1} |H|\,dx + \int_{U_4} |H|\,dx = \\ &\int_{U_1\cup U_4} |H|\,dx \leq 2\int_B \varphi(|f|)|Q|\,dx,\end{aligned}$$

and thus

$$\int_B \varphi(|P|)|Q|\,dx \leq 5\Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx \tag{2.14}$$

for $Q \in \Pi^m$ such that $Q(t)P(t) < 0$.
Finally, (2.8) follows from (2.13) and (2.14) □

The next corollary will be useful in the sequel.

Corollary 2.5. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let $f \in L^\Phi(B)$. If P is the best polynomial approximation of $f \in L^\Phi(B)$, then*

$$\int_B \varphi(|P|)|P|\,dx \leq 5\Lambda_\varphi \|P\|_\infty \int_B \varphi(|f|)\,dx. \tag{2.15}$$

Proof. It follows for $Q = P$ in (2.8) of Theorem 2.4 and employing $|P| \leq \|P\|_\infty$. □

Remark 2.6. In order to obtain Theorem 2.4 we only have used that the polynomial P is a solution of (2.7) for f in $L^\varphi(B)$. Thus the inequality (2.15) holds for any polynomial P that satisfies identity (2.7) and f belonging to $L^\varphi(B)$.

3. Extension of the best polynomial approximation to $L^\varphi(B)$

In order to get a continuous extension of $\mu_\Phi(f)$ for functions in the bigger space $L^\varphi(B)$ we need the following auxiliary results. Throughout this section we will consider $\varphi \in \Im$ and $\Phi(x) = \int_0^x \varphi(t)\,dt$.

Lemma 3.1. *Let f_n be a sequence in $L^\Phi(B)$, such that there exists a constant C that satisfies $\int_B \varphi(|f_n|)\,dx \le C$. Then $\{\|P\|_\infty : P \in \mu_\Phi(f_n), n = 1, 2, \ldots\}$ is bounded.*

Proof. Using Corollary 2.5, we have

$$\int_B \varphi(|P|)|P|\,dx \le 5\Lambda_\varphi \|P\|_\infty \int_B \varphi(|f_n|)\,dx \le 5C\Lambda_\varphi \|P\|_\infty, \tag{3.1}$$

for each $P \in \mu_\Phi(f_n)$ and for every all n. Thus, using (1.2) we get

$$\int_B \Phi(|P|)\,dx \le 5\Lambda_\varphi C \|P\|_\infty.$$

Then, from Jensen's inequality, we obtain

$$|B|\Phi\Big(\frac{1}{|B|}\int_B |P|\,dx\Big) \le \int_B \Phi(|P|)\,dx.$$

Now, since $\|P\|_1$ is a norm which is equivalent to $\|P\|_\infty$, for $P \in \Pi^m$, we obtain for a suitable constant K,

$$\Phi\Big(\frac{K}{|B|}\|P\|_\infty\Big) \le 5\Lambda_\varphi^2 \frac{C}{|B|}\|P\|_\infty.$$

Thus taking into account that $\frac{\Phi(x)}{x}$ goes to ∞ as x tends to ∞ the lemma is proved. □

Lemma 3.2. *Let f_n, f be functions in $L^\varphi(B)$ such that*

$$\int_B \varphi(|f_n - f|)\,dx \to 0 \tag{3.2}$$

as $n \to \infty$.
Also let g_n, g be measurable functions such that $|g_n| \le C$ for all n and $g_n \to g$ a.e. as $n \to \infty$. Then there exists a subsequence n_k such that

$$\int_B \varphi(|f_{n_k}|)g_{n_k}\,dx \to \int_B \varphi(|f|)g\,dx \tag{3.3}$$

as $k \to \infty$.

Proof. Since φ is a non decreasing function and $\varphi(x) > 0$ for $x > 0$, there exists a subsequence f_{n_k} which converges to f a.e. We will use now that the sequence $\varphi(|f_n|)$ has an equiabsolutely continuous integrals. That means, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$\int_E \varphi(|f_n|)\,dx \leq \varepsilon$, for any $E \subset B$, $|E| \leq \delta$, and for every n. This fact follows at once from $\int_B \varphi(|f - f_n|)\,dx \to 0$, and

$$\int_E \varphi(|f_n|) \leq \Lambda_\varphi \int_B \varphi(|f - f_n|)\,dx + \Lambda_\varphi \int_E \varphi(|f|)\,dx.$$

Now, by the Egorov's theorem, given $\delta > 0$ there exists $F \subset B$, $|B - F| < \delta$ such that the subsequence $\varphi(|f_{n_k}|)g_{n_k}$ uniformly converges to $\varphi(|f|)g$ on F. Then

$$\begin{aligned}&\int_B \varphi(|f_{n_k}|)g_{n_k}\,dx - \int_B \varphi(|f|)g\,dx = \\ &\int_{B-F} (\varphi(|f_{n_k}|)g_{n_k} - \varphi(|f|)g)\,dx + \int_F (\varphi(|f_{n_k}|)g_{n_k} - \varphi(|f|)g)\,dx \\ &= I_k + J_k.\end{aligned}$$

Now, using the uniform convergence of the sequence on F we have that J_k goes to 0 as k goes to ∞. On the other hand, since we are dealing with equiabsolutely continuous integrals we get $|I_k| < \varepsilon$ for every k. □

Theorem 3.3. *If $f \in L^\varphi(B)$, then there exists $P \in \Pi^m$ such that*

$$\int_B \varphi(|f - P|)\text{sgn}(f - P)Q\,dx = 0, \tag{3.4}$$

for every $Q \in \Pi^m$.
And

$$\int_B \Phi(|P|)\,dx \leq K\|P\|_\infty \int_B \varphi(|f|)\,dx, \tag{3.5}$$

for a suitable constant K.

Proof. Set the sequence of functions $f_n = min(max(f, -n), n)$ which are in $L^\Phi(B)$. Then, by Theorem 2.3, there exists $P_n \in \mu_\Phi(f_n)$ such that

$$\int_B \varphi(|f_n - P_n|)\text{sgn}(f_n - P_n)Q\,dx = 0, \tag{3.6}$$

for every $Q \in \Pi^m$.
Observe that $\int_B \varphi(|f_n - f|)\,dx \to 0$, as $n \to \infty$. Now, by Lemma 3.1, the sequence $\|P_n\|_\infty$ is bounded. Then there exists a subsequence P_{n_k} which uniformly converges on B to a polynomial $P \in \Pi^m$. Thus, by Lemma 3.2, we get

$$\begin{aligned}0 = \lim_{k\to\infty} &\int_B \varphi(|f_{n_k} - P_{n_k}|)\text{sgn}(f_{n_k} - P_{n_k})Q\,dx = \\ &\int_B \varphi(|f - P|)\text{sgn}(f - P)Q\,dx,\end{aligned}$$

for every $Q \in \Pi^m$.
Now, by Remark 2.6 and (1.2) we also get

$$\int_B \Phi(|P|)\,dx \le \int_B \varphi(|P|)\,|P|\,dx \le 5\Lambda_\varphi \|P\|_\infty \int_B \varphi(|f|)\,dx,$$

and the proof is completed. □

Now Theorem 3.3 allows us to extend the definition of the best approximation operator for functions in $L^\varphi(B)$.

Definition 2. *For $f \in L^\varphi(B)$ we set $\mu_\varphi(f)$ for the set of polynomials $P \in \Pi^m$ that satisfy (3.4) and we refer to this set as the extended best approximation operator.*

Next, we list some properties of this best approximation operator.

Theorem 3.4. *If Φ is a strictly convex function, then there exists a unique extended best polynomial approximation for every $f \in L^\varphi(B)$.*

Proof. For $f \in L^\varphi(B)$ we consider $P_1, P_2 \in \mu_\varphi(f)$, $P_1 \ne P_2$, then

$$\begin{aligned}\int_B \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)Q\,dx = \\ \int_B \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)Q\,dx = 0,\end{aligned} \tag{3.7}$$

for every $Q \in \Pi^m$.
Set the polynomial $R = P_1 - P_2 \in \Pi^m$ and the pairwise disjoint sets

$$\begin{aligned} A &= \{x \in B : P_2(x) > P_1(x)\} \\ B &= \{x \in B : P_1(x) > P_2(x)\} \\ C &= \{x \in B : P_1(x) = P_2(x)\} \end{aligned}$$

then $A \cup B \cup C = B$ and $\mu(C) = 0$.
Since Φ is a strictly convex function we have that $\varphi(|x|)\mathrm{sgn}(x)$ is a strictly increasing function. Consider $R < 0$ and $f - P_2 < f - P_1$ on the set A, then

$$\varphi(|f-P_2|)\mathrm{sgn}(f-P_2) < \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)$$

and thus

$$\varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R < \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R.$$

Hence

$$\int_A \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R\,dx \le \int_A \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R\,dx. \tag{3.8}$$

Analogously if $R > 0$ and $f - P_1 < f - P_2$ on the set B, then

$$\varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R < \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R.$$

Therefore

$$\int_B \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R\,dx \le \int_B \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R\,dx. \tag{3.9}$$

Now, since P_1 and P_2 are continuous functions and $P_1 \neq P_2$ on B, then $\mu(A) > 0$ or $\mu(B) > 0$. Thus, at least one of the inequalities (3.8) or (3.9) must be strict.
From (3.7), (3.8) and (3.9) we get

$$\begin{aligned}
&0 = \int_B \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R\,dx = \\
&\int_A \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R\,dx + \int_B \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R\,dx < \\
&\int_A \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R\,dx + \int_B \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R\,dx = \\
&\int_B \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R\,dx = 0,
\end{aligned}$$

which is a contradiction and the proof is completed. □

Proposition 3.5. *For any $f \in L^\varphi(B)$ it satisfies $\mu_\varphi(f+P) = \mu_\varphi(f) + P$ for all $P \in \Pi^m$.*

Proof. It follows directly from the definition of the extended best approximation operator $\mu_\varphi(f)$. □

Theorem 3.6. *Let Φ be a strictly convex function and $h_n, h \in L^\varphi(B)$ such that*

$$\int_B \varphi(|h_n - h|)\,dx \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Then $\mu_\varphi(h_n) \to \mu_\varphi(h)$ as $n \to \infty$.

Proof. Set $P_n = \mu_\varphi(h_n)$. By inequality (3.5) the sequence P_n is uniformly bounded. We consider a subsequence P_{n_k} which converges to a polynomial P. Now, we select a subsequence of h_{n_k}, which will be also called by h_{n_k}, that converges to h a.e; we also have, for any $Q \in \Pi^m$,

$$\int_B \varphi(|h_{n_k} - P_{n_k}|)\mathrm{sgn}(h_{n_k} - P_{n_k})Q\,dx = 0. \tag{3.11}$$

Now, by Lemma 3.2, we get

$$\int_B \varphi(|h - P|)\mathrm{sgn}(h - P)Q\,dx = 0, \tag{3.12}$$

and taking into account Theorem 3.4, $P = \mu_\varphi(f)$ and the whole sequence P_n converges to P. Thus the proof is completed. □

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